Nijenhuis structures on Courant algebroids

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From Tulczyjew's geometry of the tangent bundle to Courant algebroids

What do I remember from Tulczyjew's work? Many things:

- The Lagrange differential (CRAS 1975) $\longrightarrow$ the exact sequence of the calculus of variations.
- T^*TM, TT^*M and T^*T^*M.
- The Legendre transformation (1973, 1977) $\longrightarrow$ case of Lie algebroids, mechanics on Lie algebroids.
- The contravariant approach to symplectic structures (Bull. Acad. Polon. Sci., 1974)$\longrightarrow$ Poisson geometry.
- Hamiltonian field theory (with Kijowski, LNPh 107, 1979).
- Noether theorem.
- etc.

"From the geometry of the tangent bundle to Lie algebroids to Courant algebroids by way of Lie bialgebras."

Our aim is to study the infinitesimal deformations of Courant algebroids. In particular,

- double of a Lie bialgebroid,
- double of a Lie algebroid,
- generalized tangent bundle of a manifold.

Advances in the theory of Nijenhuis operators were made by

- Fuchssteiner [1997], for general algebraic structures,
- Bedjaoui-Tebbal [2000], on the study of contractions of Lie algebras,
- Cariñena, Grabowski and Marmo [2001], on the study of contractions and deformations of both Lie algebras and Leibniz (Loday) algebras,
- Cariñena, Grabowski and Marmo [2004], on the study of Leibniz algebroids, in particular Courant algebroids (henceforth [CGM]),
- Clemente-Gallardo and Nunes da Costa [2004], on the case of Courant algebroids,
- Grabowski [2006], on the supermanifold approach to Courant algebroids (henceforth [G]).

Our contribution is mainly

- a careful study of the properties of Nijenhuis tensors on Courant algebroids,
- a reformulation and new proofs with few or no computations of the results of [CGM],
- the definition and properties of the 'weak deforming tensors'.

Our description of Nijenhuis structures and related concepts relies on the use of Roytenberg's graded Poisson bracket on the minimal symplectic realization of a Courant algebroid [2002], and on its interplay with the big bracket (Roytenberg [2002], yks [1992, 2004, 2005, 2011]).

We shall argue that, in the deformation theory of a Courant structure by a skew-symmetric tensor, the decisive property is not the vanishing of the Nijenhuis torsion of the tensor but the property which we call weak deforming.

What we wish to present

- The big bracket, derived bracket approach to Lie algebras, Lie bialgebras, Lie algebroids, Lie bialgeboids.
- Leibniz (Loday) algebras, double of a Lie bialgebroid, Courant algebroids.
- Results of Cariñena-Grabowski-Marmo [2001][2004] on Nijenhuis structures on Leibniz algebras and Leibniz algebroids.
- Definition of irreducibility, adapted from Grabowski [2006].
- Properties of the Nijenhuis torsion of a skew-symmetric endomorphism of a Courant algebroid.
- Definition of 'weak deforming tensors'.
- Infinitesimal deformations of Courant structures.
- The case of the double of a Lie bialgebroid.
- The case of the double of a trivial Lie bialgebroid.
- Examples: PN-structures and ΩN-structures on a Lie algebroid define infinitesimal deformations of its double.

An even graded Poisson bracket

Kostant and Sternberg [1987]: for a vector space E equipped with a symmetric bilinear form, $\langle\ ,\ \rangle$, there is a unique graded Poisson bracket of degree -2 on $\wedge^\bullet E$, $\{\ ,\ \}$, that extends $\langle\ ,\ \rangle$ as a biderivation.

Lecomte and Roger [1990], then yks [1992] considered the case $E = V \oplus V^*$ with the canonical symmetric bilinear form.

Then $\{\ ,\ \}$ is a Poisson bracket of bidegree $(-1,-1)$ on $\wedge^\bullet(V \oplus V^*) = \oplus_{k\geq 0, \ell \geq 0} \wedge^k V \otimes \wedge^\ell V^*$, called the big bracket.

Supermanifold interpretation

$V \oplus V^*$ is the cotangent bundle T^*V of V.

Consider the supermanifold $\Pi V = V[1]$ and its cotangent bundle $T^*[2]V[1]$.
The algebra of functions on $T^*[2]V[1]$ is $\wedge^\bullet(V \oplus V^*)[1]$.

The big bracket $\{\ ,\ \}$ is the Poisson bracket of the canonical symplectic structure of this cotangent bundle.

In Lie algebra theory

In restriction to $V \otimes \wedge^{\bullet} V^*$, the even Poisson bracket $\{\ ,\ \}$ is the Nijenhuis-Richardson bracket of vector-valued forms on V,
and similarly for vector-valued forms on V^* (up to a sign).

Example 1:
A Lie algebra structure on V is an element $\mu \in V \otimes \wedge^2 V^*$ such that

$$\{\mu, \mu\} = 0.$$

$(\mu : \wedge^2 V \to V)$

Example 2:
A Lie coalgebra structure on V is an element $\gamma \in \wedge^2 V \otimes V^*$ such that

$$\{\gamma, \gamma\} = 0.$$

$(\mu : \wedge^2 V^* \to V^*)$

The Lie bracket as a derived bracket

Let (V, μ) be a Lie algebra, where μ is the Lie algebra structure.
Then $\wedge^\bullet V$ is equipped with the Schouten bracket, $[\ ,\ ]^\mu$.
Proposition Let $X,\ Y \in \wedge^\bullet V$. Then

$$\boxed{[X, Y]^\mu = \{\{X, \mu\}, Y\}}.$$

In particular,

- the Lie bracket of $X,\ Y \in V$,

$$[X, Y] = [X, Y]^\mu = \{\{X, \mu\}, Y\}\ .$$

This result shows that the Lie bracket is a derived bracket.

The cohomological approach to Lie bialgebras

A Lie bialgebra structure on a vector space V is defined by $\mu \in V \otimes \wedge^2 V^*$ and $\gamma \in \wedge^2 V \otimes V^*$ such that

$$\boxed{\{\mu + \gamma, \mu + \gamma\} = 0}.$$

$$\begin{cases} \{\mu, \mu\} = 0 \ , \\ \{\mu, \gamma\} = 0 \ , \\ \{\gamma, \gamma\} = 0 \ . \end{cases}$$

The compatibility conditon,

γ is a cocycle of (V, μ) or μ is a cocycle of (V^*, γ),

is simply expressed as

$$\{\mu, \gamma\} = 0.$$

The Lie algebra cohomology operators of (V, μ) and (V^*, γ) are $\mathrm{d}_\mu = \{\mu, .\}$ and $\mathrm{d}_\gamma = \{\gamma, .\}$: bicomplex of the Lie bialgebra.

The double of a Lie bialgebra (V, μ, γ)

The double of a Lie bialgebra (V, μ, γ) is the Lie algebra

$$\boxed{(V \oplus V^*, \mu + \gamma)}.$$

Generalizations

• Lie algebra $\longrightarrow$ Lie algebroid.

• Lie bialgebra $\longrightarrow$ Lie bialgebroid
Mackenzie and Xu [1994] (infinitesimal of a Poisson groupoid), yks [1995].

• double of a Lie bialgebra $\longrightarrow$ double of a Lie bialgebroid
(in particular, generalized tangent bundle)
Liu-Weinstein-Xu [1997].

• double of a Lie bialgebroid $\longrightarrow$ Courant algebroid.
Liu-Weinstein-Xu [1997].

The bigraded algebra $\mathcal{F}$

When $A \to M$ is a vector bundle,
let $A[1]$ be the graded manifold obtained from A by assigning
- degree 0 to the coordinates on the base,
- degree 1 to the coordinates on the fibers.

Let $\mathcal{F}$ be the bigraded commutative algebra of smooth functions on $T^*[2]A[1]$.

Remark When M is a point, A is just a vector space.
Then $\mathcal{F} = \wedge^\bullet(A \oplus A^*)[1]$. See above.

Local coordinates on $T^*[2]A[1]$, and their bidegrees:

$$\begin{array}{cccc} x^i & \tau^\alpha & p_i & \theta_\alpha \\ (0,0) & (0,1) & (1,1) & (1,0) \end{array}$$

The big bracket for vector bundles

As the cotangent bundle of a graded manifold, $T^*[2]A[1]$ is canonically equipped with an even Poisson structure.
Denote the even Poisson bracket on $\mathcal{F}$ by $\{\ ,\ \}$.
It is also called the big bracket because in the case of a vector bundle over a point (*i.e.*, a vector sapce) it reduces to the big bracket we have defined.
It is of bidegree $(-1,-1)$.

- It is skew-symmetric, $\{u,v\} = -(-1)^{|u|\,|v|}\{v,u\}$, $u,\ v \in \mathcal{F}$,
- It satisfies the Jacobi identity,

$$\{u,\{v,w\}\} = \{\{u,v\},w\} + (-1)^{|u|\,|v|}\{v,\{u,w\}\}\ ,$$

$u,\ v,\ w \in \mathcal{F}$.
In local coordinates,

$$\boxed{\{x^i,p_j\} = \delta^i_j \quad \text{and} \quad \{\tau^\alpha,\theta_\beta\} = \delta^\alpha_\beta}.$$

Lie algebroids

A Lie algebroid structure on $A \to M$ is an element μ of $\mathcal{F}$ of bidegree $(1,2)$ such that

$$\{\mu, \mu\} = 0 \ .$$

Schouten bracket of multivectors (sections of $\wedge^\bullet A$) X and Y:

$$[X, Y] = \{\{X, \mu\}, Y\}, \qquad X, Y \in \Gamma \wedge^\bullet A.$$

In particular,

• the Lie bracket of $X,\ Y \in \Gamma A$,

$$[X, Y] = \{\{X, \mu\}, Y\} \ .$$

• the anchor of A, $\rho : A \to TM$,

$$\rho(X)f = [X, f] = \{\{X, \mu\}, f\} \ ,$$

for $X \in \Gamma A$, $f \in C^\infty(M)$.

ρ induces a Lie algebra homomorphism from ΓA to $\Gamma(TM)$.

Examples of Lie algebroids

- TM for M any manifold,
- Lie algebra (Lie algebroid over a point),
- T^*M when M is a Poisson manifold,
- action Lie algebroids,
- etc.

In local coordinates,

$$\mu = \rho^i_\alpha \tau^\alpha p_i - \frac{1}{2} C^\gamma_{\alpha\beta} \tau^\alpha \tau^\beta \theta_\gamma .$$

The differential of a Lie algebroid

The operator $\mathrm{d} = \{\mu, .\}$ is a differential on $\Gamma(\wedge^\bullet A^*)$ which defines the Lie algebroid cohomology of A.

Lie algebroid cohomology generalizes
• Chevalley–Eilenberg cohomology (when M is a point, A is a Lie algebra), and
• de Rham cohomology (when $A = TM$).

The Lichnerowicz–Poisson cohomology of a Poisson manifold is another example.

Lie bialgebroids

A Lie bialgebroid structure on a vector bundle $A \to M$ is defined by μ of bidegree $(1,2)$ and γ of bidegree $(2,1)$ such that

$$\{\mu + \gamma, \mu + \gamma\} = 0.$$

Fact The double $V \oplus V^*$ of a Lie bialgebra is a Lie algebra.

Questions

- Is the double $A \oplus A^*$ of a Lie bialgebroid a Lie algebroid?
- In particular, is $TM \oplus T^*M$ a Lie algebroid?
- Is $\mathcal{X}(M) \oplus \Omega^1(M)$ a Lie algebra?

Before we consider the double of a Lie bialgebroid, we shall define Leibniz (Loday) algebras.

Leibniz algebras

A *Leibniz algebra* (or *Loday algebra*) is a vector space L over a field k of characteristic 0, equipped with a k-bilinear composition law, called the *Leibniz bracket*, satisfying the Jacobi identity,

$$\boxed{u \circ (v \circ w) = (u \circ v) \circ w + v \circ (u \circ w)},$$

for all u, v, w in L.

A Leibniz algebra with a skew-symmetric composition law is a Lie algebra.

The *Leibniz cohomology*, defined by Loday [1993], is a generalization of the Chevalley-Eilenberg cohomology of Lie algebras.

Graded Leibniz algebra,
$u \circ (v \circ w) = (u \circ v) \circ w + (-1)^{|u||v|} v \circ (u \circ w)$.

Derived bracket by an interior derivation

Theorem

Let S be an odd element in a graded Lie algebra $(L, \{\ ,\ \})$ such that $\{S, S\} = 0$. Then

$$[u, v] = \{\{\ u, S\}, v\}$$

defines a graded Leibniz bracket on $L[1]$.

(This theorem is a particular case of the main result of yks [1996].)

The double of a Lie bialgebroid

Let (A, μ, γ) be a Lie bialgebroid. Consider the Dorfman bracket on $A \oplus A^*$ defined by

$$\boxed{[u, v]_D = \{\{u, \mu + \gamma\}, v\}},$$

for u and $v \in \Gamma(A \oplus A^*)$.

By the preceding theorem, it is a Leibniz bracket.

The skew-symmetrized Dorfman bracket is called the Courant bracket.

If $\gamma = 0$, the Lie bialgebroid is called *trivial*.

Generalized tangent bundles

In particular, if $A = TM$, equipped with the identity endomorphism as anchor and the Lie bracket of vector fields, and if $\gamma = 0$, then $TM \oplus T^*M$ is the *standard Courant algebroid* or *generalized tangent bundle* of M.

The Dorfman bracket on $TM \oplus T^*M$ is explicitly

$$\boxed{[X + \xi + \eta]_D = [X, Y] + \mathcal{L}_X \eta - i_Y(\mathrm{d}\xi)},$$

for all vector fields, X and Y, and all 1-forms, ξ and η, on M.

The original bracket of T. Courant [1990] is recovered as the skew-symmetrized Dorfman bracket.

Towards the definition of Courant algebroids

We follow the approach of Roytenberg [2002].

Let $(E, \langle\ ,\ \rangle)$ be a vector bundle over a manifold M, equipped with a fiberwise symmetric bilinear form.

Let j be the injective map fron $E[1]$ to $E[1] \oplus E^*[1]$ defined by

$$u \mapsto (u, \frac{1}{2}\langle u, \cdot\rangle).$$

The map $j : E[1] \to E[1] \oplus E^*[1]$ is such that $< ju, jv >= \langle u, v\rangle$ for all $u, v \in E$, where $<\ ,\ >$ is the canonical fiberwise symmetric bilinear form on $E \oplus E^*$.

The minimal symplectic realization of $(E, \langle\ ,\ \rangle)$ is the bundle

$$\boxed{\widetilde{E} = j^!(T^*[2]E[1])},$$

where $j^!$ denotes the pull-back by j.

The graded Poisson bracket $\{\ ,\ \}$

The algebra of functions on the minimal symplectic realization $\widetilde{E}$ of E is canonically equipped with a graded Poisson bracket of degree -2.

We denote this Poisson bracket by $\{\ ,\ \}$ and this graded Poisson algebra by $\mathcal{A}$.

Facts

$\mathcal{A}^0 = C^\infty(M)$, $\mathcal{A}^1 = \Gamma E$.

For all sections u, v of E,

$$\{u, v\} = \langle u, v\rangle.$$

Local coordinates Let (e_a) be a local basis of sections of E and let $g_{ab} = \langle e_a, e_b\rangle$.

Let $(q^i, \tau^a, p_i, \theta_a)$ be canonical local coordinates on $T^*[2]E[1]$.

If $\theta_a = \frac{1}{2} g_{ab}\tau^b$, then (q^i, τ^a, p_i) are local coordinates on $\widetilde{E}$ such that

$$\{q^i, p_j\} = \delta^i_j \quad \text{and} \quad \{\tau^a, \tau^b\} = g^{ab}.$$

Definition of Courant algebroids

A *Courant algebroid* structure on a vector bundle $(E, \langle\ ,\ \rangle)$ over a manifold M, equipped with a fiberwise symmetric bilinear form, is defined by an element $\Theta \in \mathcal{A}^3$ such that

$$\boxed{\{\Theta, \Theta\} = 0}.$$

We shall assume that $\langle\ ,\ \rangle$ is non-degenerate.

Example The double of a Lie bialgebroid with $\Theta = \mu + \gamma$.

In local coordinates,

$$\Theta = \rho_a^i \tau^a p_i - \frac{1}{6} C_{abc} \tau^a \tau^b \tau^c.$$

The anchor and bracket as derived brackets

In a Courant algebroid defined by Θ,

• the *anchor* $\rho : E \to TM$ is defined by $\boxed{\rho(u)f = \{\{u, \Theta\}, f\}}$,

• the *bracket* of sections of E is defined by $\boxed{[u, v] = \{\{u, \Theta\}, v\}}$,

for all sections u, v of E, and $f \in C^\infty(M)$.

As in the case of the double of a Lie bialgebroid,

- bracket [,] is called the *Dorfman bracket*.
- the *Courant bracket* is the skew-symmetrization of the Dorfman bracket.

The bracket of sections is the derived bracket by Θ of the canonical Poisson bracket of $\mathcal{A}$ restricted to $\mathcal{A}^1 \times \mathcal{A}^1 = \Gamma E \times \Gamma E$, therefore the Dorfman bracket is a Leibniz bracket.

We denote a Courant algebroid by $(E, \langle\ ,\ \rangle, \Theta)$, or just E.

The operator $\mathrm{d}_\Theta = \{\Theta, \cdot\}$ is a cohomology operator on $\mathcal{A}$.

Two relations

Define $\partial : C^\infty(M) \to \Gamma E$ by

$$\langle u, \partial f \rangle = \rho(u) \cdot f.$$

We shall make use of the relations,

$$[u, v] + [v, u] = \partial \langle u, v \rangle,$$

and

$$\langle [u, v], w \rangle + \langle v, [u, w] \rangle = \langle u, \partial \langle v, w \rangle \rangle.$$

Tensors on E and functions on $\widetilde{E}$

Let $(E, \langle\ ,\ \rangle)$ be a vector bundle equipped with a fiberwise symmetric bilinear form.
Any skew-symmetric contravariant or covariant tensor, t, can be identified with a function $\widetilde{t}$ on $\widetilde{E}$.

In local coordinates, to a skew-symmetric contravariant k-tensor, $t = t^{a_1 a_2 \dots a_k} e_{a_1} e_{a_2} \dots e_{a_k}$, corresponds

$$\widetilde{t} = \frac{1}{k!} t^{a_1 a_2 \dots a_k} g_{a_1 b_1} g_{a_2 b_2} \dots g_{a_k b_k} \tau^{b_1} \tau^{b_2} \dots \tau^{b_k} \in \mathcal{A}^k.$$

When no confusion can arise, we shall sometimes write t for $\widetilde{t}$.
A section $u = u^a e_a$ of E is identified with the function $u = g_{ab} u^a \tau^b \in \mathcal{A}^1$.

Skew-symmetric endomorphisms

Let $\mathcal{N} : E \to E$ be a skew-symmetric endomorphism of the vector bundle E, *i.e.*, $\mathcal{N}$ is such that

$$\mathcal{N} + {}^t\mathcal{N} = 0.$$

Remark Other authors call these endomorphisms *orthogonal*.

In local coordinates, if $\mathcal{N}(e_a) = \mathcal{N}_a^b e_b$, the skew-symmetry condition is

$$\mathcal{N}_a^b g_{bc} = \mathcal{N}_c^b g_{ba}.$$

Then

$$\widetilde{\mathcal{N}} = \frac{1}{2}\mathcal{N}_a^b g_{bc} \tau^a \tau^c \in \mathcal{A}^2.$$

Proposition

In terms of the Poisson bracket of $\mathcal{A}$, $\boxed{\mathcal{N}(u) = \{u, \widetilde{\mathcal{N}}\}}$, *for all sections* u *of* E.

Let $(L, \circ)$ be a Leibniz algebra. For an endomorphism N of L, define

$$\boxed{u \circ_N v = Nu \circ v + u \circ Nv - N(u \circ v)},$$

and set

$$\boxed{(TN)(u, v) = Nu \circ Nv - N(u \circ_N v)}.$$

$TN : L \times L \to L$ is called the *Nijenhuis torsion* or simply the *torsion* of N,
N is said to be a *Nijenhuis tensor* or a *Nijenhuis structure* on $(L, \circ)$ if $TN = 0$.

Deformations of Leibniz brackets [CGM]

Proposition

A necessary and sufficient condition for $\circ_N$ to be a Leibniz bracket is that TN be a Leibniz cocycle.

Proposition

When N is a Nijenhuis tensor on $(L, \circ)$,
(i) $\circ_N$ is a Leibniz bracket,
(ii) N is a morphism of Leibniz algebras from $(L, \circ_N)$ to $(L, \circ)$, and
(iii) $\circ_N$ is compatible with $\circ$ in the sense that their sum is a Leibniz bracket.

I.e., Nijenhuis tensors define trivial infinitesimal deformations of Leibniz brackets.

Bracket $[\ ,\ ]_{\mathcal{N}}$

Let $(E, \langle\ ,\ \rangle, \Theta)$ be a Courant algebroid over a manifold M, where $\langle\ ,\ \rangle$ is a fiberwise symmetric bilinear form. The function $\Theta \in \mathcal{A}^3$ determines the anchor, ρ, and the Leibniz bracket on sections, $[\ ,\ ]$. We shall assume that $\mathcal{N} : E \to E$ is a skew-symmetric vector bundle endomorphism. We define, for all $u, v \in \Gamma E$,

$$\boxed{[u, v]_{\mathcal{N}} = [\mathcal{N}u, v] + [u, \mathcal{N}v] - \mathcal{N}[u, v]}.$$

Proposition

In terms of the Poisson bracket of $\mathcal{A}$,

$$\boxed{[u, v]_{\mathcal{N}} = \{\{u, \{\widetilde{\mathcal{N}}, \Theta\}\}, v\}},$$

for all $u, v \in \Gamma E = \mathcal{A}^1$.

Nijenhuis torsion

We now define the *Nijenhuis torsion*, or simply the *torsion* of $\mathcal{N}$ by

$$\boxed{(T_\Theta \mathcal{N})(u,v) = [\mathcal{N}u, \mathcal{N}v] - \mathcal{N}[u,v]_\mathcal{N}},$$

for all sections u, v of E. A skew-symmetric endomorphism whose torsion vanishes is called a *Nijenhuis tensor.*

Remark For an endomorphism that is not skew-symmetric, the torsion can still be defined by the same formula. Many results are also valid for endomorphisms $\mathcal{N}' = \mathcal{N} + \kappa\,\mathrm{Id}_E$, which are characterized by the condition $\mathcal{N}' + {}^t\mathcal{N}' = 2\kappa\,\mathrm{Id}_E$. Such endomorphisms are called *paired* in [CGM].

Use of the Courant bracket

One can also define the torsion $T_C\mathcal{N}$ of an endomorphism $\mathcal{N}$ with respect to the Courant bracket, replacing the Dorfman bracket by its skew-symmetrization in the preceding formulas. The relation between the two torsions is

$$(T_C\mathcal{N})(u,v) = \frac{1}{2}\left((T_\Theta\mathcal{N})(u,v) - (T_\Theta\mathcal{N})(v,u)\right).$$

Proposition

For a skew-symmetric endomorphism $\mathcal{N}$,

$$\begin{aligned} & (T_C\mathcal{N})(u,v) - (T_\Theta\mathcal{N})(u,v) \\ = \; & \tfrac{1}{2}\left(\partial\langle u, \mathcal{N}^2 v\rangle - \mathcal{N}^2\partial\langle u,v\rangle\right). \end{aligned}$$

If $\mathcal{N}^2$ is a scalar multiple of the identity of E, both torsions, $T_\Theta\mathcal{N}$ and $T_C\mathcal{N}$, coincide.

Almost cps and cps structures

We introduce the following definition, due to I. Vaisman, where 'cps' stands for 'complex, product or subtangent'.

Definition

An endomorphism $\mathcal{N}$ of a vector bundle E such that $\mathcal{N}^2 = \lambda \, \mathrm{Id}_E$, where $\lambda = -1$, 0 or 1, is called an *almost cps structure* on E. An almost cps structure on a Lie algebroid or a Courant algebroid is called a *cps structure* if its torsion vanishes.

When $(E, \langle \ , \ \rangle, \Theta)$ is a Courant algebroid, the torsion $T_\Theta \mathcal{N}$ of $\mathcal{N} : E \to E$ is a map from $\Gamma E \times \Gamma E$ to ΓE. Unlike the case of Lie algebroids, $T_\Theta \mathcal{N}$ is not in general $C^\infty(M)$-linear in both arguments, and in general it is not skew-symmetric.

Properties of the torsion. Lack of C^∞-linearity and of skew-symmetry

Let $\mathcal{N}$ be a skew-symmetric endomorphism of a Courant algebroid.

Lack of C^∞-linearity.
It is clear that

$$(T_\Theta \mathcal{N})(u, fv) = f(T_\Theta \mathcal{N})(u, v),$$

but

$$(T_\Theta \mathcal{N})(fu, v) = f(T_\Theta \mathcal{N})(u, v) + \langle u, v\rangle \mathcal{N}^2(\partial f) - \langle u, \mathcal{N}^2 v\rangle \partial f.$$

Lack of skew-symmetry.
Using the fact that $\mathcal{N}$ is skew-symmetric, we obtain

$$(T_\Theta \mathcal{N})(u, v) + (T_\Theta \mathcal{N})(v, u) = \mathcal{N}^2 \partial \langle u, v\rangle - \partial \langle u, \mathcal{N}^2 v\rangle.$$

Associated 3-tensor

In order to determine whether $T_\Theta\mathcal{N}$ determines a skew-symmetric covariant 3-tensor, we use the skew-symmetry of $\mathcal{N}$ and relations stated above to obtain

$$\langle (T_\Theta\mathcal{N})(u,v),w\rangle + \langle (T_\Theta\mathcal{N})(u,w),v\rangle = \langle \mathcal{N}^2[u,w] - [u,\mathcal{N}^2 w],v\rangle.$$

Theorem When $\mathcal{N}^2 = \lambda\,\mathrm{Id}_E$, the torsion of $\mathcal{N}$ is $C^\infty(M)$-linear in both arguments and skew-symmetric, and defines a skew-symmetric covariant 3-tensor on E, $\widetilde{T_\Theta\mathcal{N}} \in \mathcal{A}^3$, by

$$\widetilde{T_\Theta\mathcal{N}}(u,v,w) = \langle (T_\Theta\mathcal{N})(u,v),w\rangle.$$

.../...

Torsion of almost cps structures

Theorem (continued)
When $\mathcal{N}^2 = \lambda \, \mathrm{Id}_E$,
(i) For all sections u, v of E,

$$\boxed{(T_\Theta \mathcal{N})(u, v) = \{\{u, \widetilde{T_\Theta \mathcal{N}}\}, v\}}.$$

(ii)

$$\boxed{\widetilde{T_\Theta \mathcal{N}} = -\frac{1}{2}(\{\{\widetilde{\mathcal{N}}, \Theta\}, \widetilde{\mathcal{N}}\} + \lambda \Theta)}.$$

Nijenhuis tensors on irreducible Courant algebroids

Definition

A Courant algebroid $(E, \langle\ ,\ \rangle, \Theta)$ is *irreducible* if any symmetric vector bundle endomorphism ϕ of E such that, for all sections u and v of E,

$$[u, \phi v] = \phi[u, v] \quad \text{and} \quad [\phi u, v] = \phi[u, v],$$

is a scalar multiple of the identity endomorphism, Id_E, of E.

Remark This definition is inspired by, but different from Grabowski's definition in [G]. Any irreducible Courant algebroid in the sense of Grabowski is irreducible in this sense.

Theorem

If E is irreducible, any Nijenhuis tensor on E is proportional to a cps structure.

An equivalent result was proved in [CGM].

Deforming tensors

Tensors with vanishing Nijenhuis torsion do not, in general, define trivial infinitesimal deformations of the Dorfman bracket of a Courant algebroid unless they have additional properties such as proportionality to an almost cps structure. We are thus led to introduce the following definitions.

Definition

A skew-symmetric endomorphism $\mathcal{N}$ of a Courant algebroid $(E, \langle \ , \ \rangle, \Theta)$ is called a
(i) *weak deforming tensor* for Θ if $\{\{\widetilde{\mathcal{N}}, \Theta\}, \widetilde{\mathcal{N}}\}$ is a d_Θ-cocycle,
(ii) *deforming tensor* for Θ if $\{\{\widetilde{\mathcal{N}}, \Theta\}, \widetilde{\mathcal{N}}\}$ is a scalar multiple of Θ.

This terminology is justified by the fact that, because

$$\mathrm{d}_\Theta \Theta = \{\Theta, \Theta\} = 0,$$

any deforming tensor is a weak deforming tensor.
The next theorem is further justification for the terms that we have introduced.

Deformations of Courant algebroids

Theorem

Let $\mathcal{N}$ be a skew-symmetric endomorphism of a Courant algebroid $(E, \langle\ ,\ \rangle, \Theta)$. Then $\{\widetilde{\mathcal{N}}, \Theta\}$ is a Courant algebroid structure on $(E, \langle\ ,\ \rangle)$ if and only if $\mathcal{N}$ is a weak deforming tensor for Θ.

Proof Use the Jacobi identity to show that

$$\{\{\widetilde{\mathcal{N}}, \Theta\}, \{\widetilde{\mathcal{N}}, \Theta\}\} = \{\Theta, \{\{\widetilde{\mathcal{N}}, \Theta\}, \widetilde{\mathcal{N}}\}\}.$$

When $\mathcal{N}$ is a weak deforming tensor, the Courant algebroid structure $\{\widetilde{\mathcal{N}}, \Theta\}$ on $(E, \langle\ ,\ \rangle)$ is compatible with Θ. In fact, $\{\Theta + \{\widetilde{\mathcal{N}}, \Theta\}, \Theta + \{\widetilde{\mathcal{N}}, \Theta\}\}$ vanishes.

Weak Nijenhuis tensors

The condition $\{\Theta, T_\Theta \mathcal{N}\} = 0$ makes sense only if $T_\Theta \mathcal{N}$ is an element of $\mathcal{A}^3$. If E is irreducible this is the case if and only if $\mathcal{N}$ is proportional to an almost cps structure, whence the following definition.

Definition

If $\mathcal{N}$ is proportional to an almost cps structure, and if $T_\Theta \mathcal{N}$ is a d_Θ-cocycle, $\mathcal{N}$ is called a *weak Nijenhuis tensor*.

Implications and equivalence

In the special case where tensors are proportional to an almost cps structure,

• A Nijenhuis tensor is a weak Nijenhuis tensor.

• A Nijenhuis tensor is a deforming tensor, and therefore also a weak deforming tensor.

• A tensor is weak Nijenhuis if and only if it is weak deforming.

$$\begin{array}{ccc} \text{Nijenhuis} & \Rightarrow & \text{weak Nijenhuis} \\ \Downarrow & & \Updownarrow \\ \text{deforming} & \Rightarrow & \text{weak deforming} \end{array}$$

Corollary

Let $\mathcal{N}$ be a skew-symmetric endomorphism of a Courant algebroid $(E, \langle\ ,\ \rangle, \Theta)$, proportional to an almost cps structure. Then $\{\widetilde{\mathcal{N}}, \Theta\}$ is a Courant algebroid structure on $(E, \langle\ ,\ \rangle)$ if and only if $\mathcal{N}$ is a weak Nijenhuis tensor.

Morphism property of $\mathcal{N}$

While the compatibility of Θ and $\{\widetilde{\mathcal{N}}, \Theta\}$ is satisfied as soon as $\mathcal{N}$ is weak deforming, it is the vanishing of the torsion which implies a morphism property of $\mathcal{N}$. Recall that an almost cps structure is a cps structure if and only if its torsion vanishes.

Proposition

Let $\mathcal{N}$ be a skew-symmetric endomorphism of E proportional to an almost cps structure. Then $\mathcal{N}$ is a morphism of Courant algebroids from $(E, \langle \ , \ \rangle, \{\widetilde{\mathcal{N}}, \Theta\})$ to $(E, \langle \ , \ \rangle, \Theta)$ if and only if $\mathcal{N}$ is proportional to a cps structure.

Cf. also [CGM] and [G].

The case of the double of a Lie bialgebroid

Let A be a vector bundle and let $E = A \oplus A^*$ be equipped with the canonical symmetric bilinear form $\langle\ ,\ \rangle$. The minimal symplectic realization of E is $\widetilde{E} = T^*[2]A[1]$. Then $\mathcal{A} = \mathcal{F}$, and the canonical Poisson bracket of $\mathcal{A}$ coincides with the big bracket.

Almost cps structures on $A \oplus A^*$

Any vector bundle endomorphism of $E = A \oplus A^*$ is of the form $\mathcal{N} = \begin{pmatrix} N & \pi \\ \omega & N' \end{pmatrix}$, where $N : A \to A$, $N' : A^* \to A^*$, $\pi : A^* \to A$ and $\omega : A \to A^*$.

The endomorphism $\mathcal{N}$ is skew-symmetric if and only if $N' = -{}^tN$, π is a bivector on A, and ω is a 2-form on A.

The conditions for $\mathcal{N}^2 = \lambda\, \mathrm{Id}_E$ are (i) $N\pi$ is a bivector, (ii) ωN is a 2-form and (iii) $N^2 + \pi\omega = \lambda\, \mathrm{Id}_A$.

N, $\mathcal{N}$ and $\widetilde{\mathcal{N}}$

A vector bundle endomorphism N of A, considered as an element in $A^* \otimes A$, defines a skew-symmetric endomorphism
$\mathcal{N} = \begin{pmatrix} N & 0 \\ 0 & -{}^tN \end{pmatrix}$ of $A \oplus A^*$.
Then $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ satisfy

$$\mathcal{N}(X + \xi) = \{X + \xi, \widetilde{\mathcal{N}}\},$$

for all $X \in \Gamma A$ and $\xi \in \Gamma(A^*)$.

In local coordinates

Let (e_α) be a local basis of sections of A and let (ϵ^α) be the dual basis. Let $(x^i, \tau^\alpha, p_i, \theta_\alpha)$ be local coordinates on $T^*[2]A[1]$. If $N = N^\alpha_\beta \epsilon^\beta e_\alpha$, then $\widetilde{\mathcal{N}} = N^\alpha_\beta \tau^\beta \theta_\alpha$.
As elements in $\mathcal{A}^2$, N and $\widetilde{\mathcal{N}}$ are identified.

Question

Consider a Lie bialgebroid (A, μ, γ).

Question Compare $T_{\mu+\gamma}\mathcal{N}$ with the torsions $T_\mu N$ and $T_\gamma\, {}^tN$?

Answer It is the sum of the two in a suitble sense.

Tensors on A and functions on $T^*[2]A[1]$

To a tensor $t \in A \otimes \wedge^2 A^*$ we associate $\widetilde{t}$ in $\wedge^3(A \oplus A^*)$ such that

$$\widetilde{t}(X+\xi, Y+\eta, Z+\zeta) = \langle t(X,Y),\zeta\rangle + \langle t(Y,Z),\xi\rangle + \langle t(Z,X),\eta\rangle,$$

for all $X, Y, Z \in A$ and $\xi, \eta, \zeta \in A^*$. The map induced on sections of $A \oplus A^*$ by $\widetilde{t}$ is the element in $\mathcal{A}$ that corresponds to t. Then

$$\widetilde{t}(X+\xi, Y+\eta, Z+\zeta) = \{\{\{X+\xi, \widetilde{t}\}, Y+\eta\}, Z+\zeta\}.$$

In local coordinates

If $t = \frac{1}{2} t^{\alpha}_{\beta\gamma} \epsilon^{\beta}\epsilon^{\gamma} e_{\alpha}$, then $\widetilde{t} = \frac{1}{2} t^{\alpha}_{\beta\gamma}\tau^{\beta}\tau^{\gamma}\theta_{\alpha}$.

There is a similar definition of $\widetilde{s}$ for $s \in \wedge^2 A \otimes A^*$.

Torsion of $\mathcal{N}$ in the case of the double of a Lie bialgebroid

When (A, μ) and (A^*, γ) are Lie algebroids, if the torsion $T_{\mu+\gamma}\mathcal{N}$ of $\mathcal{N}$ defines an element in $\mathcal{A}^3$, we can compare $\widetilde{T_{\mu+\gamma}\mathcal{N}}$ with the sum of the elements in $\Gamma(\wedge^3(A \oplus A^*)) \subset \mathcal{A}^3$ defined by the torsion $T_\mu N$ of N and the torsion $T_\gamma\, {}^tN$ of ${}^tN$.

Theorem

Let $((A, \mu), (A^*, \gamma))$ be a Lie bialgebroid. Let $N : A \to A$ be a vector bundle endomorphism, and let $\mathcal{N}$ be the skew-symmetric endomorphism of $A \oplus A^*$ with matrix $\begin{pmatrix} N & 0 \\ 0 & -\,{}^tN \end{pmatrix}$.

(i) The element $\{\{\widetilde{\mathcal{N}}, \mu+\gamma\}, \widetilde{\mathcal{N}}\}$ is in $\mathcal{A}^3$ and is equal to $\{\{N, \mu+\gamma\}, N\}$.

(ii) If N is proportional to an almost cps structure on A, then

$$\boxed{\widetilde{T_{\mu+\gamma}\mathcal{N}} = \widetilde{T_\mu N} + \widetilde{T_\gamma\, {}^tN}}.$$

Explicit form of equation $\widetilde{T_{\mu+\gamma}\mathcal{N}} = \widetilde{T_\mu N} + \widetilde{T_\gamma\, {}^tN}$

The explicit form of equation $\widetilde{T_{\mu+\gamma}\mathcal{N}} = \widetilde{T_\mu N} + \widetilde{T_\gamma\, {}^tN}$ is

$$
\begin{aligned}
&(T_{\mu+\gamma}\mathcal{N})(X+\xi, Y+\eta, Z+\zeta) \\
&= (T_\mu N)(X, Y, \zeta) + (T_\mu N)(Y, Z, \xi) + (T_\mu N)(Z, X, \eta) \\
&+ (T_\gamma\, {}^tN)(\xi, \eta, Z) + (T_\gamma\, {}^tN)(\eta, \zeta, X) + (T_\gamma\, {}^tN)(\zeta, \xi, Y),
\end{aligned}
$$

for all sections $X+\xi, Y+\eta, Z+\zeta$ of $A \oplus A^*$.

Deformations of Lie bialgebroids

The preceding theorem implies that if $N^2 = \lambda \mathrm{Id}_A$ and both N and ${}^tN$ are Nijenhuis tensors, then $\mathcal{N} = \begin{pmatrix} N & 0 \\ 0 & -{}^tN \end{pmatrix}$, is a Nijenhuis tensor for the double of the Lie bialgebroid.

In addition, deforming (A, μ) by N and (A^*, γ) by ${}^tN$ yields a Lie bialgebroid whose double is the Courant algebroid $(A \oplus A^*, \{\widetilde{\mathcal{N}}, \mu + \gamma\})$.

The case of trivial Lie bialgebroids

We now consider the particular case of the trivial Lie bialgebroids, such as the generalized tangent bundles.
If $((A, \mu), (A^*, 0))$ is the trivial Lie bialgebroid associated with the Lie algebroid (A, μ), then

$$\widetilde{T_\mu \mathcal{N}} = \widetilde{T_\mu N}.$$

In particular, in this case, if $N^2 = \lambda \mathrm{Id}_A$, deforming the Dorfman bracket of the double by $\mathcal{N}$ amounts to deforming (A, μ) by N.

Deformations of trivial Lie bialgebroids

In the following theorem it is not required that N^2 be a scalar multiple of the identity.

Theorem

Let (A, μ) be a Lie algebroid, and let N be a vector bundle endomorphism of A. If N is a Nijenhuis tensor for (A, μ), then $\mathcal{N} = \begin{pmatrix} N & 0 \\ 0 & -{}^tN \end{pmatrix}$ is a weak deforming tensor for the Courant algebroid $(A \oplus A^*, \mu)$, and $\{\widetilde{\mathcal{N}}, \mu\}$ is a Courant algebroid structure on $A \oplus A^*$, which is the double of the trivial Lie bialgebroid defined by (A, μ_N) , where $\mu_N = \{N, \mu\}$.

Example: PN-structures and deforming tensors

Various types of composite structures on Lie algebroids give rise to infinitesimal deformations of the Dorfman bracket of the double of a trivial Lie bialgebroid. We assume that (A, μ) is a Lie algebroid, and we consider the trivial Lie bialgebroid $((A, \mu), (A^*, 0))$,

Proposition

Let N be a vector bundle endomorphism of A, and let π be a bivector on A such that $N\pi = \pi\, {}^tN$. If (π, N) is a PN-structure on A, then the skew-symmetric endomorphism of $A \oplus A^*$,
$\mathcal{N} = \begin{pmatrix} N & \pi \\ 0 & -\,{}^tN \end{pmatrix}$*, is a weak deforming tensor for $(A \oplus A^*, \mu)$.*

As a consequence we recover the well-known fact that when (π, N) is a PN-structure on A, then $\{\widetilde{\mathcal{N}}, \mu\} = \{N, \mu\} + \{\pi, \mu\}$ is a Courant algebroid structure on $A \oplus A^*$, the double of the Lie bialgebroid $((A, \mu_N), (A^*, \gamma_\pi))$, where $\gamma_\pi = \{\pi, \mu\}$. See yks-Rubtsov [2010].

PN-structures where N^2 is proportional to the identity

If N^2 is proportional to the identity of A, and if π is a bivector such that $\underbrace{N\pi = \pi\, {}^tN}$, then $\mathcal{N}^2$ is proportional to the identity of $A \oplus A^*$, and $\widetilde{T_\mu(\mathcal{N})}$ is identified with $T_\mu(N) - \frac{1}{2}[\pi,\pi]^\mu + \frac{1}{2} C_\mu(\pi, N)$ in $\mathcal{A}^3$. Using the bigrading of $\mathcal{A}$, we conclude,

Proposition

If N is proportional to an almost cps structure on A, and π is a bivector such that $N\pi = \pi\, {}^tN$, then $T_\mu(\mathcal{N}) = 0$ if and only if (π, N) is a PN-structure.

Example: ΩN-structures and deforming tensors

We can also relate ΩN-structures to deforming tensors, although there is no obvious analogue of the previous proposition.

Proposition

Let N be a vector bundle endomorphism of A, and let ω be a 2-form on A such that $\omega N = {}^tN\omega$. If (ω, N) is an ΩN-structure on A, then the skew-symmetric endomorphism of $A \oplus A^*$, $\mathcal{N} = \begin{pmatrix} N & 0 \\ \omega & -{}^tN \end{pmatrix}$, is a weak deforming tensor for $(A \oplus A^*, \mu)$.

To be continued...

•Related work in progress :

- Paulo Antunes, Camille Laurent-Gengoux and Joana Nunes da Costa on compatible structrues on Courant algebroids.
- Mathieu Stiénon and Wei Hong on holomorphic symplectic and hypercomplex structures.

•Further problems:

- Investigate the role of the Nijenhuis tensors and define Nijenhuis relations in the theory of Dirac pairs on general Courant algebroids (Dirac pairs generalize the bi-hamiltonian structures and have applications to integrable systems.)
- Develop the theory of Dirac-Nijenhuis structures ([CGM], Clemente-Gallardo and Nunes da Costa, He and Liu).

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