Variational derivation of Dixon's equations.

The goal of this talk is:

To offer a generalized Hamiltonian procedure ending in the fairly general Dixon's [1] system of the first order ordinary differential equations

$$\begin{cases} P'_n = -\frac{1}{2} R_{nm}{}^{kl} \dot{x}^m S_{kl} \\ S'_{nm} = P_n \dot{x}_m - P_m \dot{x}_n \end{cases} \qquad S_{nm} + S_{mn} = 0\,, \qquad \text{(D)}$$

In the theory of General Relativity these equations should hold true along the world line of a quasi-classical particle endowed with the inner angular momentum (said "spin") S_{nm}.

We show to the end of this talk that (D) follows from a fairly general setting of the second order parameter-ambivalent (also called *parameter-invariant*) variational problem as its Hamiltonian counterpart by the appropriate definition of S_{mn}.

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The Legendre transformation.

The coordinates in the space of the fourth-order Ehresmann velocities T^4M, $M \ni x = \{x^i\}$ are:

$$u = \dot{x} = \frac{dx}{d\tau}(0), \quad \dot{u} = \frac{d^2x}{d\tau^2}(0), \quad \ddot{u} = \frac{d^3x}{d\tau^3}(0), \quad \dddot{u} = \frac{d^4x}{d\tau^4}(0).$$

Recall the definition of the Liouville form on $T^*(TM)$:

$$\Lambda = p.dx + p^{(1)}.du\,,$$

and the definition of the Legendre transformation,

$$Le\colon T^3M \to T^*(TM) \quad \text{over} \quad TM$$

$$\wp^{(1)} \stackrel{def}{=} p^{(1)} \circ Le = \frac{\partial L}{\partial \dot{u}}\,, \tag{1.1}$$

$$\wp \stackrel{def}{=} p \circ Le = \frac{\partial L}{\partial u} - D_\tau \wp^{(1)}\,, \tag{1.2}$$

where the total derivative $\quad D_\tau = u\dfrac{\partial}{\partial x} + \dot{u}\dfrac{\partial}{\partial u} + \ddot{u}\dfrac{\partial}{\partial \dot{u}}$

Zermelo conditions.

A function $L(x, u, \dot{u})$ defined on T^2M, constitutes a parameter-ambivalent variational problem $\delta \int L d\tau = 0$ if and only if it satisfies the Zermelo conditions:

$$u^i \frac{\partial L}{\partial \dot{u}^i} \equiv 0 \tag{2.1}$$

$$u^i \frac{\partial L}{\partial u^i} + 2\,\dot{u}^i \frac{\partial L}{\partial \dot{u}^i} - L \equiv 0\,. \tag{2.2}$$

In terms of the Legendre transformation the Zermelo conditions give rise to the following:

$$Z_1 \stackrel{\text{def}}{=} u^\alpha {\wp^{(1)}}_\alpha = 0 \tag{3.1}$$

$$Z \stackrel{\text{def}}{=} u^\alpha \wp_\alpha + \dot{u}^\alpha {\wp^{(1)}}_\alpha - \mathcal{L} = 0\,. \tag{3.2}$$

Generalized canonical equations.

Following Grässer [2], Rund [3], and Weyssenhoff [4], we present the system of generalized canonical equations for the second order autonomous and parameter-ambivalent variational problem in the form

$$\frac{dx}{d\tau} = \lambda \frac{\partial H}{\partial p} \circ Le \tag{G1}$$

$$\frac{du}{d\tau} = \lambda \frac{\partial H}{\partial p^{(1)}} \circ Le + \mu u \tag{G2}$$

$$\frac{dp}{d\tau} \circ Le = -\lambda \frac{\partial H}{\partial x} \circ Le \tag{G3}$$

$$\frac{dp^{(1)}}{d\tau} \circ Le = -\lambda \frac{\partial H}{\partial u} \circ Le - \mu \wp^{(1)} \tag{G4}$$

where the functions λ and μ are some undetermined multipliers. Equations (G1–G4) follow from the exterior differential equation

$$Le^{-1} i_X d\Lambda = -\lambda Le^{-1} dH - \mu dZ_1 \,.$$

Implementing covariant derivatives.

The following change of variables in the configuration space

$$\{x^n, u^n, \dot{u}^n\} \xrightarrow{\phi} \{x^n, u^n, u'^n\}, \qquad u'^n = \dot{u}^n + \Gamma^n{}_{lm} u^m u^l\,,$$

along with the introduction of the covariant momenta variables

$$\pi^1 = \frac{\partial \mathcal{L}}{\partial u'}\,, \qquad \pi = \frac{\partial \mathcal{L}}{\partial u} - \pi^{1\prime}\,, \qquad \mathcal{L} = L \circ \phi^{-1}\,,$$

suggests the corresponding change of variables in the phase space:

$$\varphi : \begin{cases} \pi^1{}_n & = p^{(1)}{}_n \\ \pi_n & = p_n - \Gamma^l{}_{mn} u^m p^{(1)}{}_l \end{cases}$$

which we derive from the Legendre transformation.
Denote:

$$\mathcal{H} = H \circ \varphi^{-1}.$$

Proposition 1. *Let $\mathcal{H}$ depend on (x, u, π, π^1) through the quontities $\gamma = u \cdot u$, $\eta = \pi^1 \cdot \pi^1$, $\psi = \pi.u$, $\nu = \pi^1.u$. In this case the Hamilton equations (G1–G4) take the shape*

$$\frac{dx}{d\tau} = u \qquad (\mathcal{H}\ 1)$$

$$u'^n = \left(\frac{\partial \mathcal{H}}{\partial \psi}\right)^{-1} \left[2\frac{\partial \mathcal{H}}{\partial \eta} g^{nk} {\pi^1}_k + \frac{\partial \mathcal{H}}{\partial \nu} u^n\right] + \mu u^n \qquad (\mathcal{H}\ 2)$$

$$\pi'_n = -R_{nkm}{}^l u^m u^k {\pi^1}_l \qquad (*)$$

$$\pi^{1\prime}{}_n = -\left(\frac{\partial \mathcal{H}}{\partial \psi}\right)^{-1} \left[2\frac{\partial \mathcal{H}}{\partial \eta} u_n + \frac{\partial \mathcal{H}}{\partial \nu} {\pi^1}_n\right] - \pi_n - \mu {\pi^1}_n \qquad (\mathcal{H}\ 4)$$

We introduced shortcuts

$$\pi'_n = \frac{d\pi_n}{d\tau} - \Gamma^m{}_{ln} \pi_m u^l, \quad \pi^{1\prime}{}_n = \frac{d\pi^1_n}{d\tau} - \Gamma^m{}_{ln} \pi^1_m u^l$$

Remark. We had no need of any special Lagrange function as far!

Proposition 2. *Equations (D) follow from (*) if we put*

$$S = u \wedge \pi^1.$$

Remark. The Mathisson supplementary conditon [5]

$$u^m S_{mn} = 0$$

is satisfied automatically.

Applications

Self-radiation.

In 1946 Bopp [6] introduced within the framework of *Special Relativity* a Lagrange function in *flat space-time* that may be expressed in terms of the geometric quantity k, the first Frenet curvature of the *radiating particle's* world line:

$$L^k = (k^2 + A)\|u\|. \qquad \text{(Bopp)}$$

Proposition 3. *In the pseudo-Riemannian space-time by means of the Legendre transformation the Lagrange function (Bopp) gives birth to the following Hamilton function for the generalized canonical formalism (G1–G4) of Grässer–Rund–Weissenhoff [2, 3, 4]*

$$\mathcal{H} = \pi.u + \frac{\|u\|^3}{4}\pi^1 \cdot \pi^1 - A\|u\|.$$

Zitterbewegung.

Proposition 4. *In the pseudo-Riemannian space-time the Lagrange function (Bopp) gives rise to the following variational equation of the type (*):*

$$\frac{D}{d\tau}\left[\Big(-3\,u'\cdot u' + A\Big)\,u_n - 2\,u''{}_n\right] = -\pi^1{}_l\, R_{nkm}{}^l u^m u^k\,, \quad (**)$$
$$u\cdot u = 1\,.$$

Proposition 5. *In* flat space-time *and on the constrained manifold*

$$k = k_0$$

the equation (**) reduces to the Riewe–Costantelos [7, 8] equation

$$\dddot{u} + \left(k^2 - \frac{m^2}{\sigma^2}\right)\dot{u} = 0, \qquad u\cdot u = 1$$

of the quasi-classical "zitterbewegung". by putting

$$A = {k_0}^2 + 2\,\frac{m^2}{\sigma^2}\,.$$

Remark. In [9] Pavšič considered in flat space-time the Hamilton function with an additional term,

$$\mathcal{H} = \pi.u + \frac{\|u\|^3}{4}\pi^1 \cdot \pi^1 + B\,\pi^1.u - A\|u\|\,.$$

It was claimed to originate from the variational problem

$$\delta\int(\tilde{k}^2 + A)\|u\|d\tau = 0, \quad \tilde{k}^2 = \frac{d^2x}{d\tau^2}\cdot\frac{d^2x}{d\tau^2}\,,$$

which I believe is not parameter-invariant.

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Thank you!