The family of finite subgroups $\mathcal{F}_M$ of a discrete group *G* associated to an action of this group over a cocompact manifold *M*

Equivariant bordisms of cocompact manifolds with action of a discrete group

An action

$$G \times M \longrightarrow M$$

of a discrete group G over a manifold is said to be proper if, for every pair of compact sets

$$K_1, K_2$$

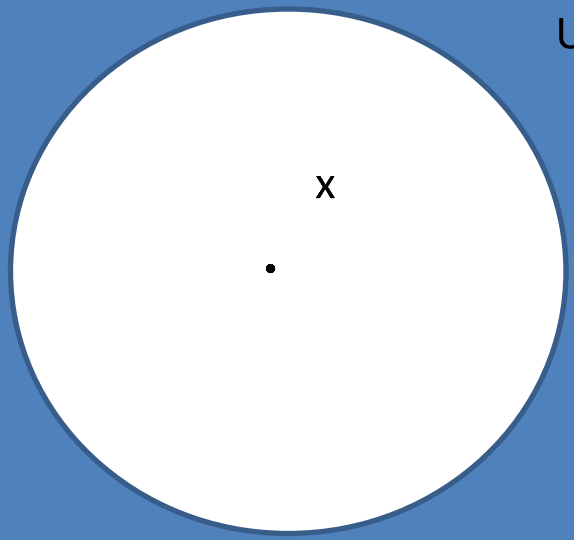
the set

$$((K_1, K_2)) = \{g \in G |\ gK_2 \cap K_1 \neq \emptyset\}$$

is finite. In particular, this means that all the isotropy groups of points are finite, and that the orbit of small neigborhoods around a given point do not accumulate.

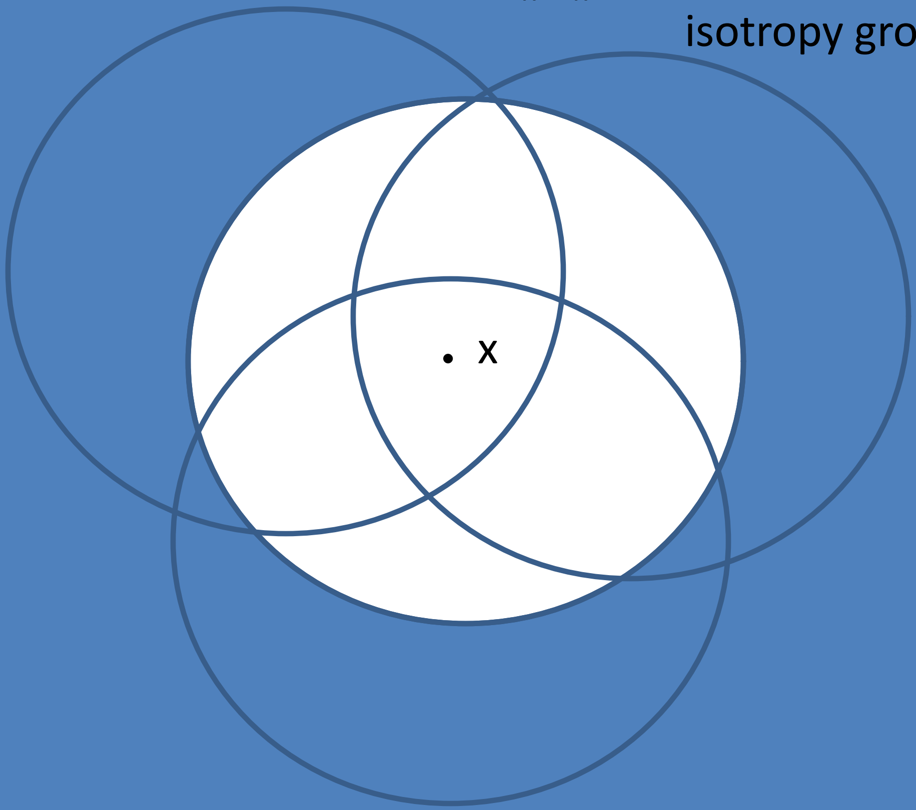
G_x is the isotropy group of the point x

U_x a neighborhood of the point x



G_x is the isotropy group of the point x

$G_x U_x$ the orbit under the action of the isotropy group



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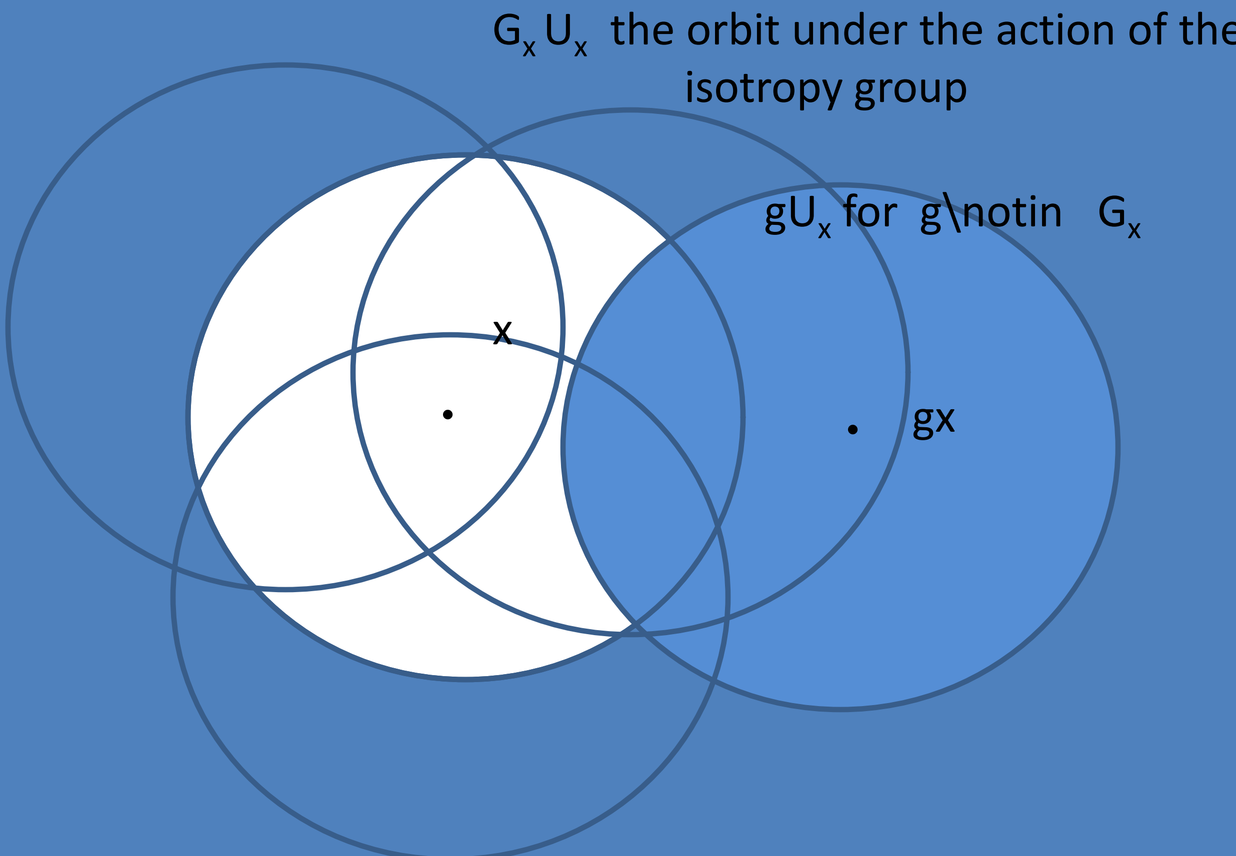

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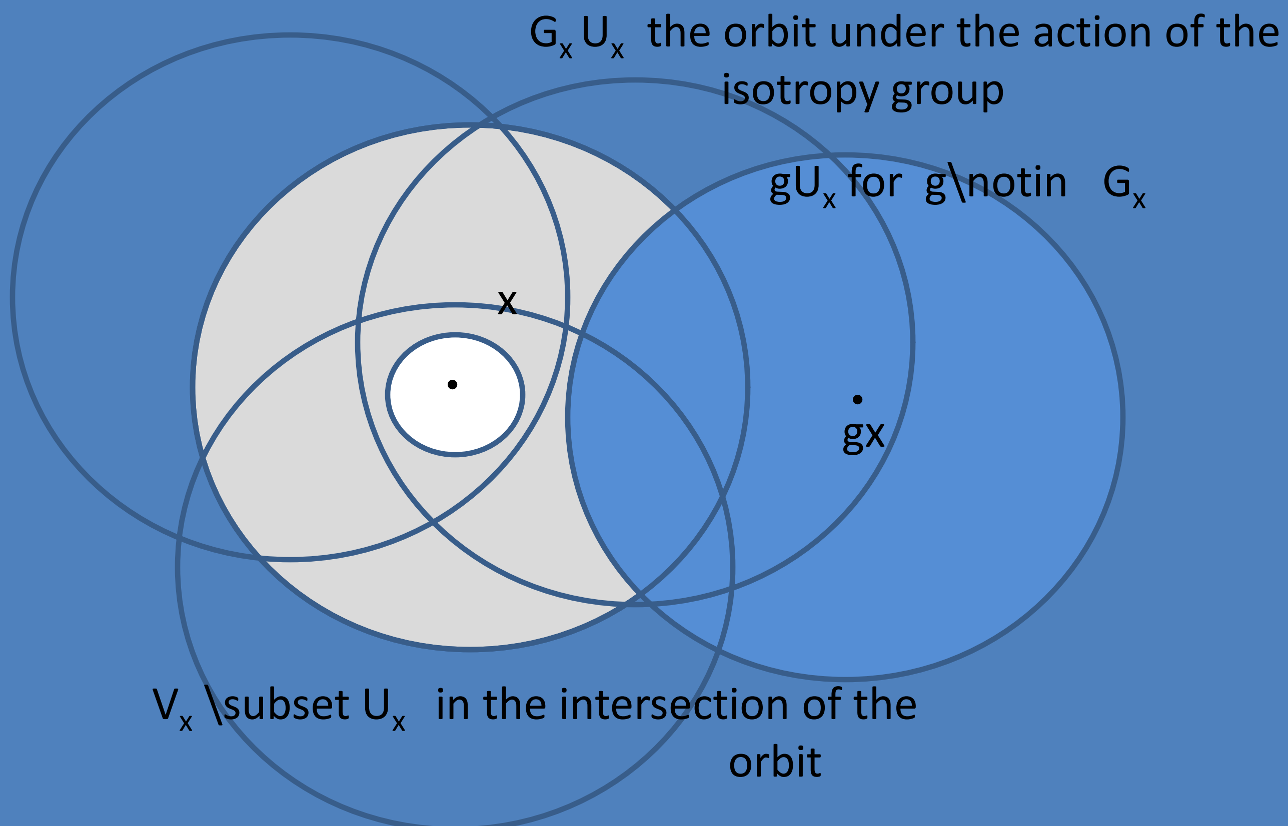

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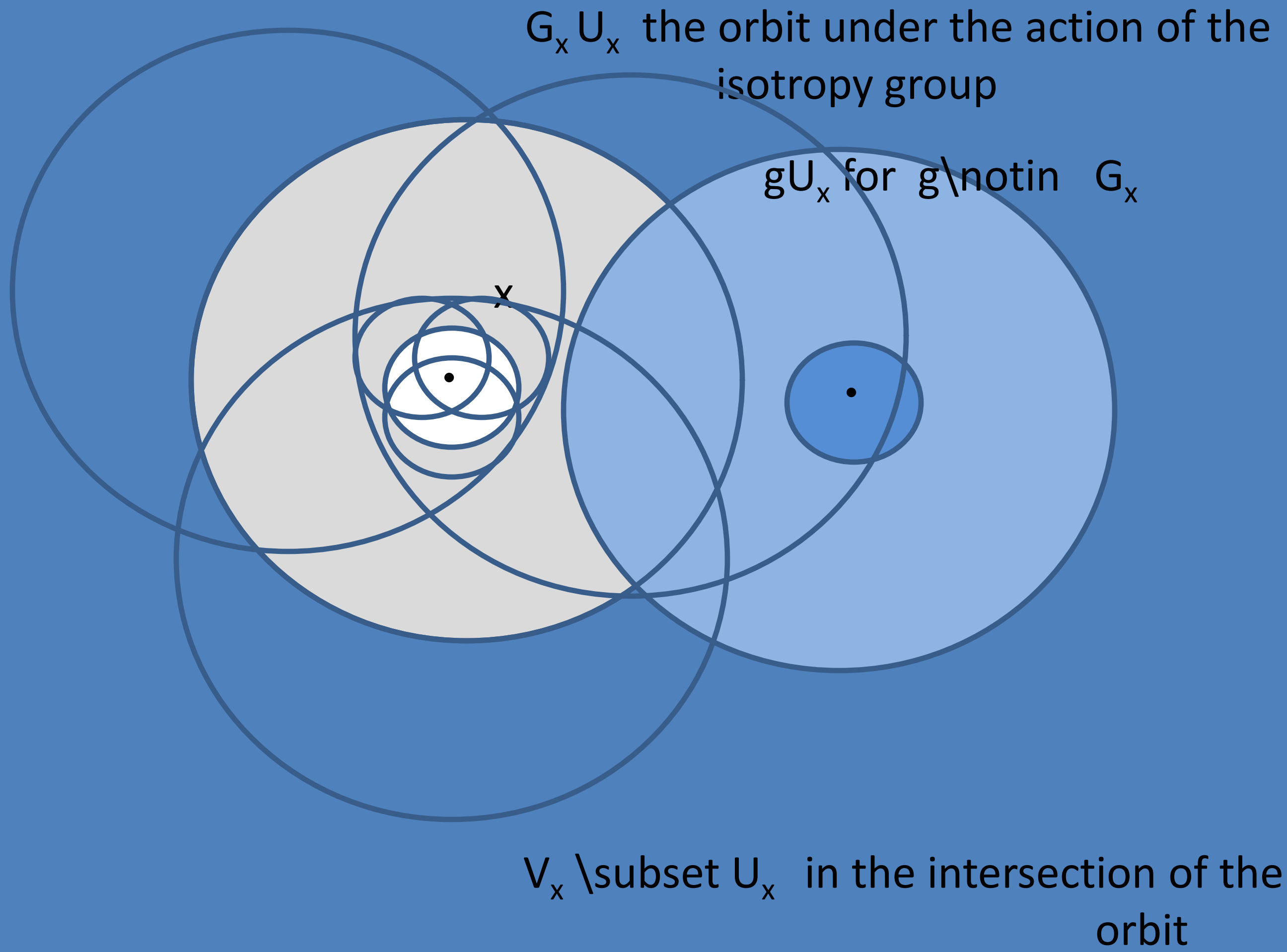

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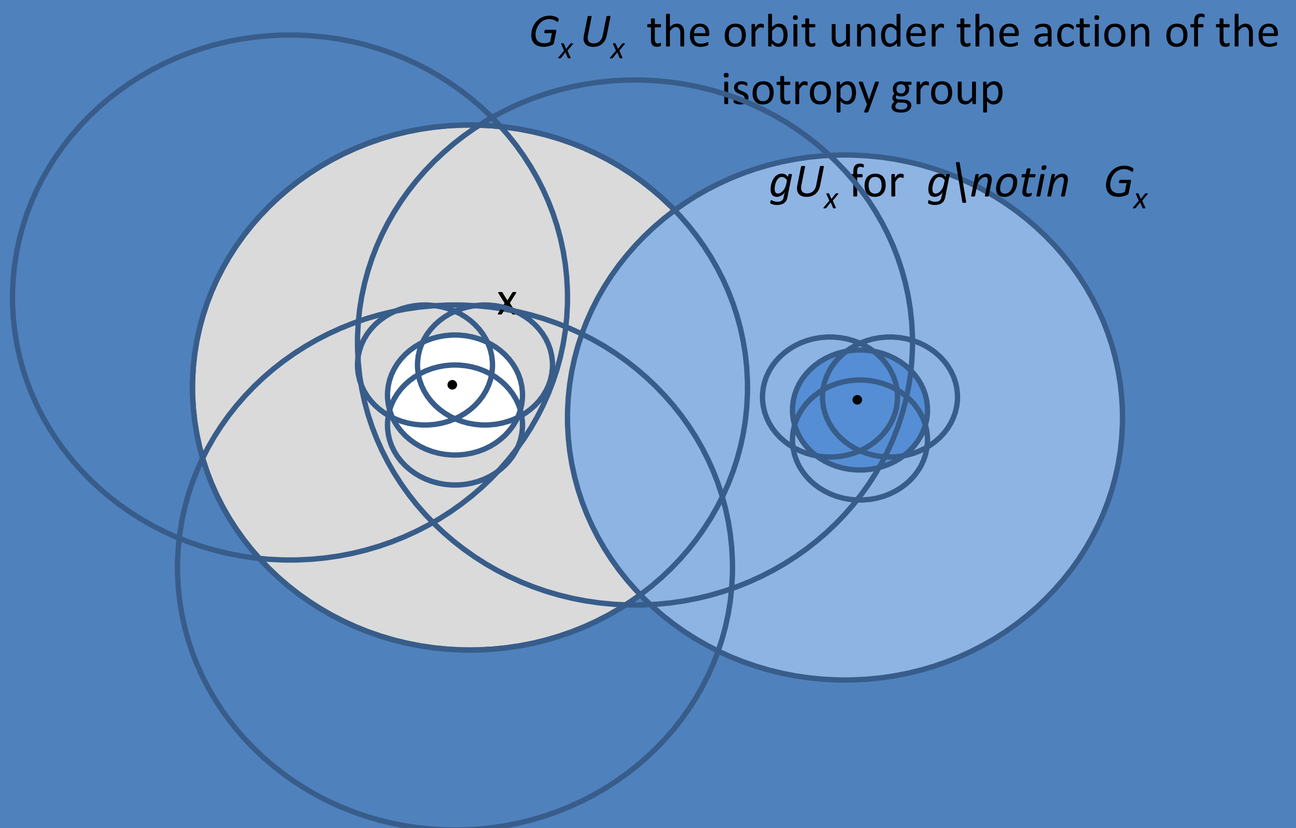
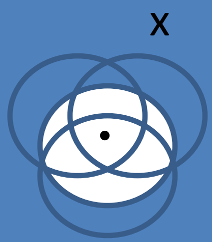


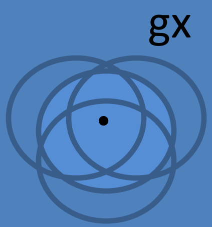
So, the *g* is not in the isoptropy group, then the translate *gU* is far enough from *U*

G_x is the isotropy group of the point x

$G_x U_x$ the orbit under the action of the isotropy group

gU_x for $g \notin G_x$

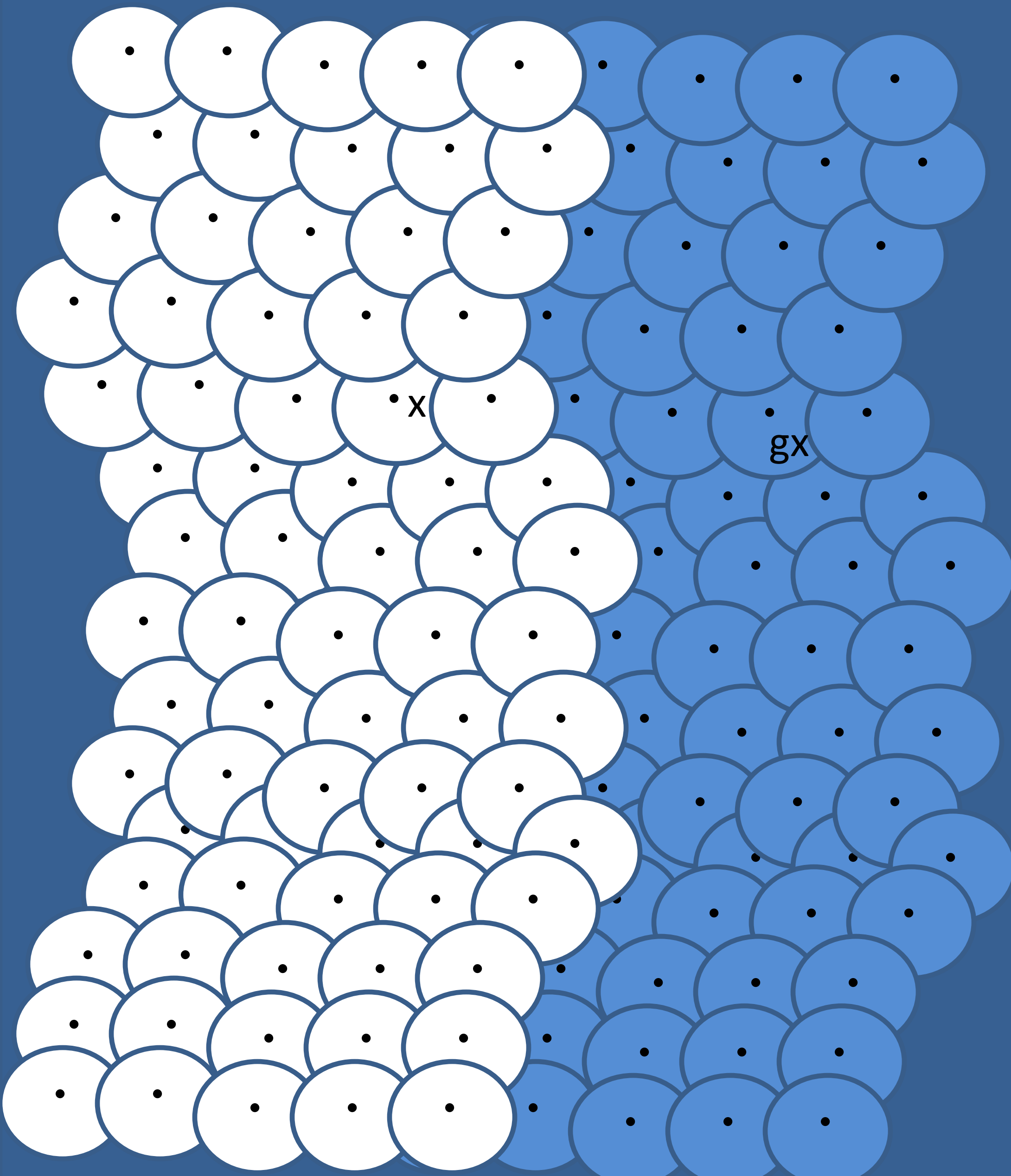




And the intersection of hU_x over $h \in G_x$ is invariant under the action of G_x so, the whole orbit GG_x is open and invariant, i.e. it projects to an open set under de factor map

$$M \rightarrow M/G$$

And this means that there is a finite covering of such neighborhoods.



And a) every such a neighborhood has atached an unique, up to conjugation, finite group, b) every element *g* of the group *G* that fixes a point *x* has to be contained in one of these groups (because it leaves invariant the neigborhood containing that particular point *x*).

In conclusion, the family of subgroups of G having non-empty set of fixed points is finite, up to group conjugation.

The Connor-Floyd fix-points construction for the case of a proper action

- Equivariant bordisms of cocompact manifolds with action of a discrete group

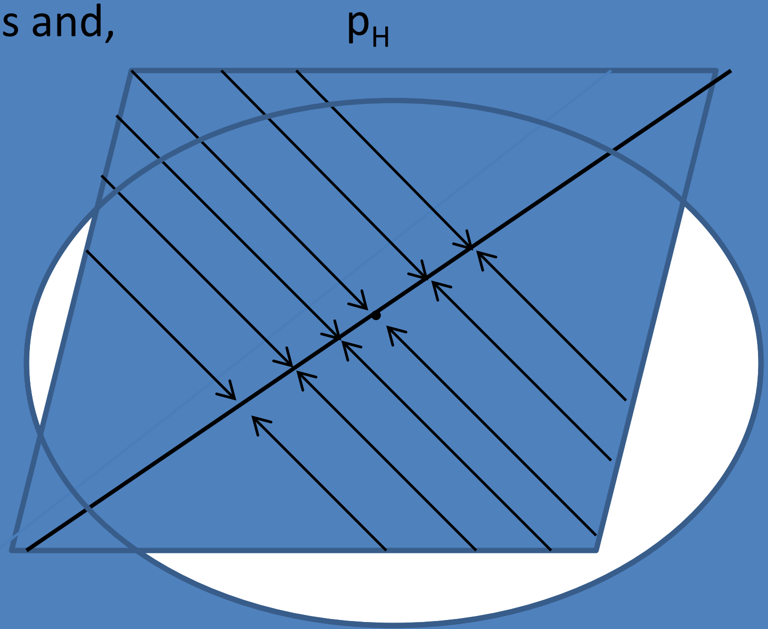
If a group H fixes a point x in M, then there is a representation \rho: H --> U(T_x M) and the operator

$$p_H = \sum_{h \in H} d_x h$$

is a projector, i.e. $\rho^2 = \rho$

and $d_x h \rho = \rho$

This means that the fixed points over the tangent space are linear subspaces and, therefore, submanifolds.



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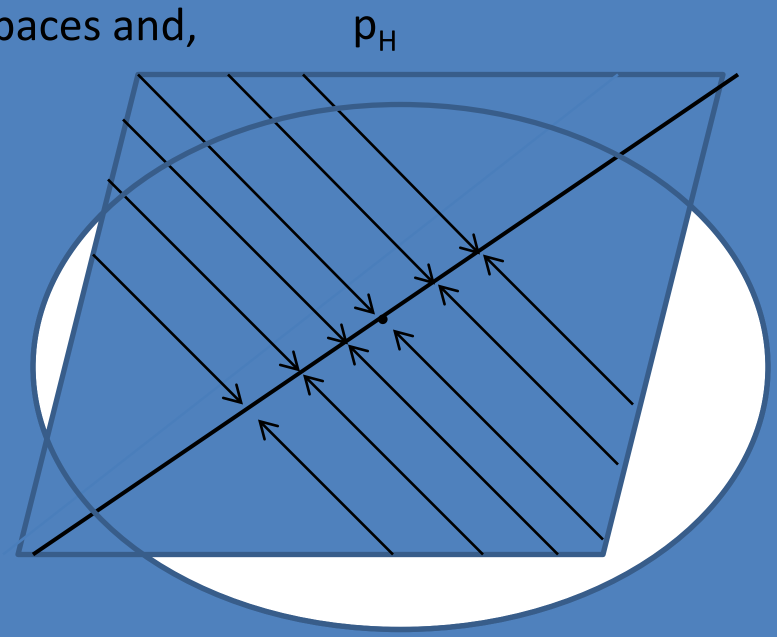
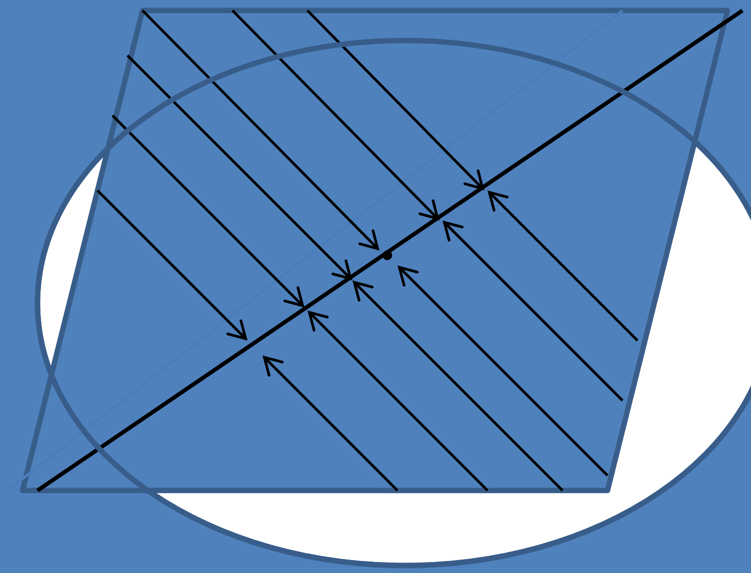


Further, we can use the exponential map (there is an invariant metric, because H is finite) to show that the set of fixed points over for the gives subgroup H constitutes a submanifold.



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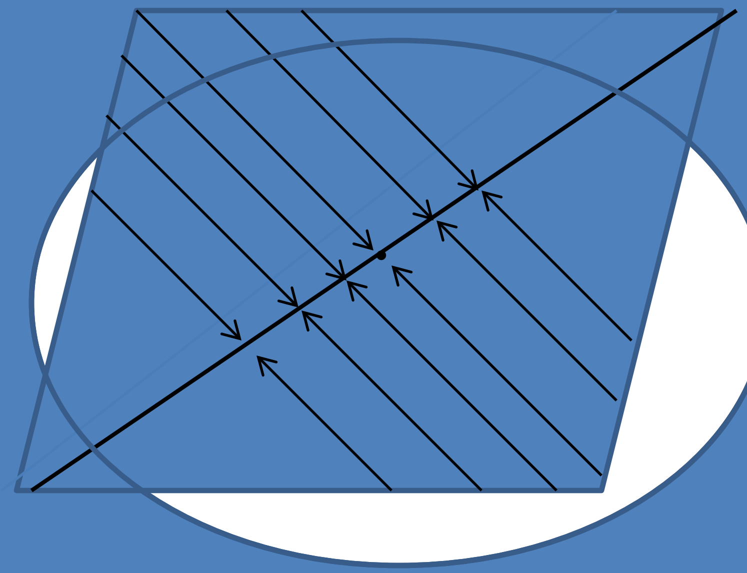

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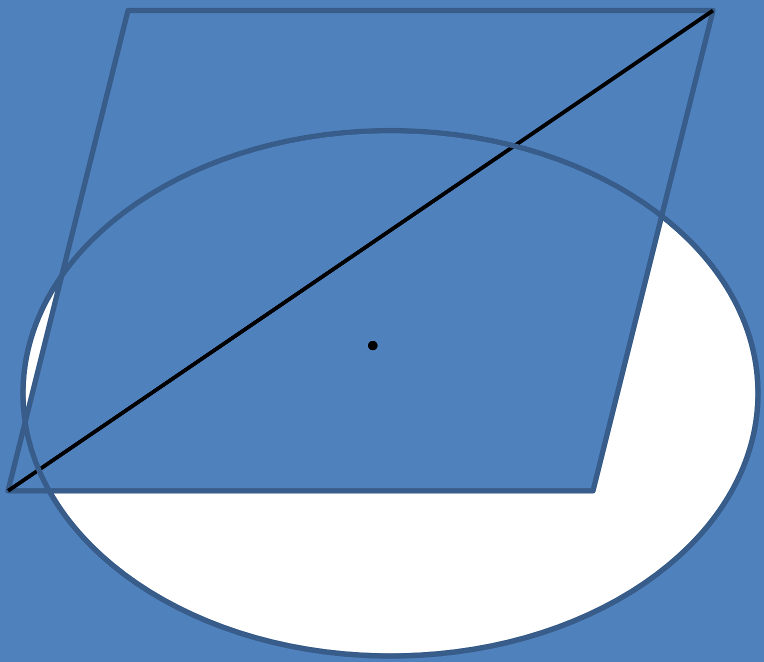

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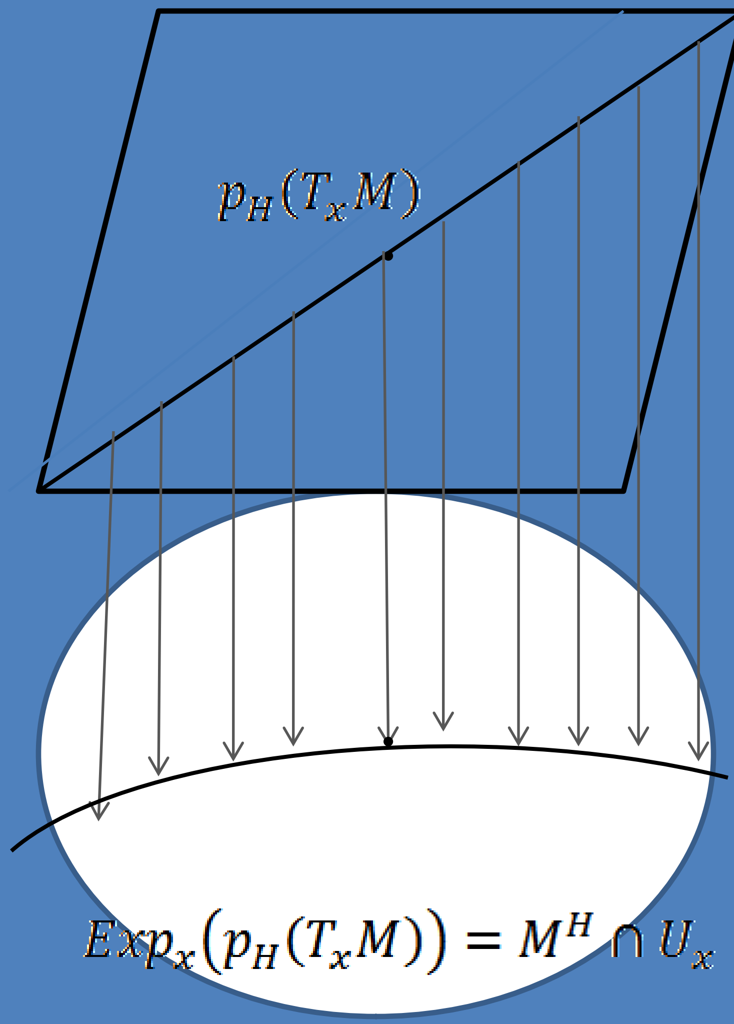

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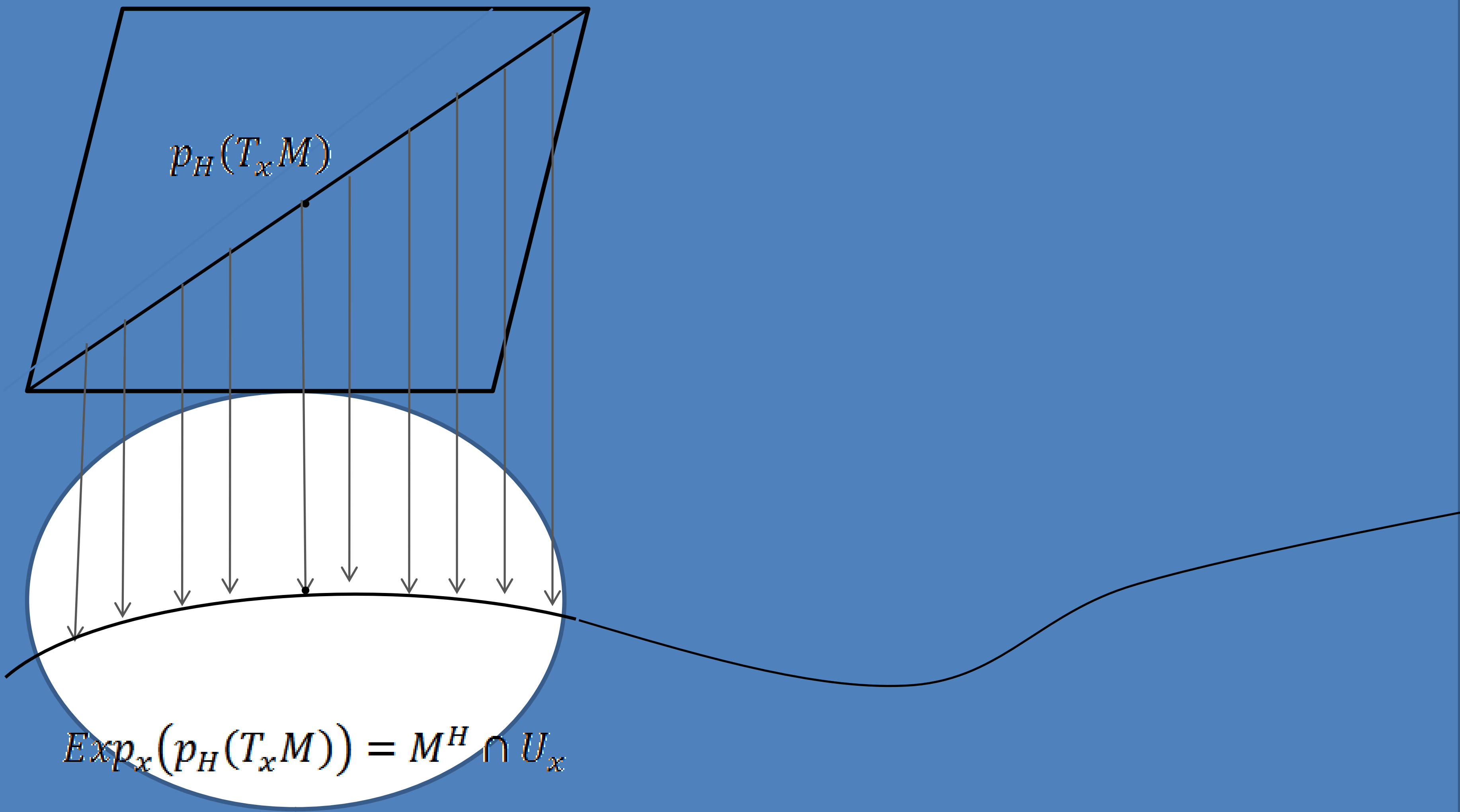
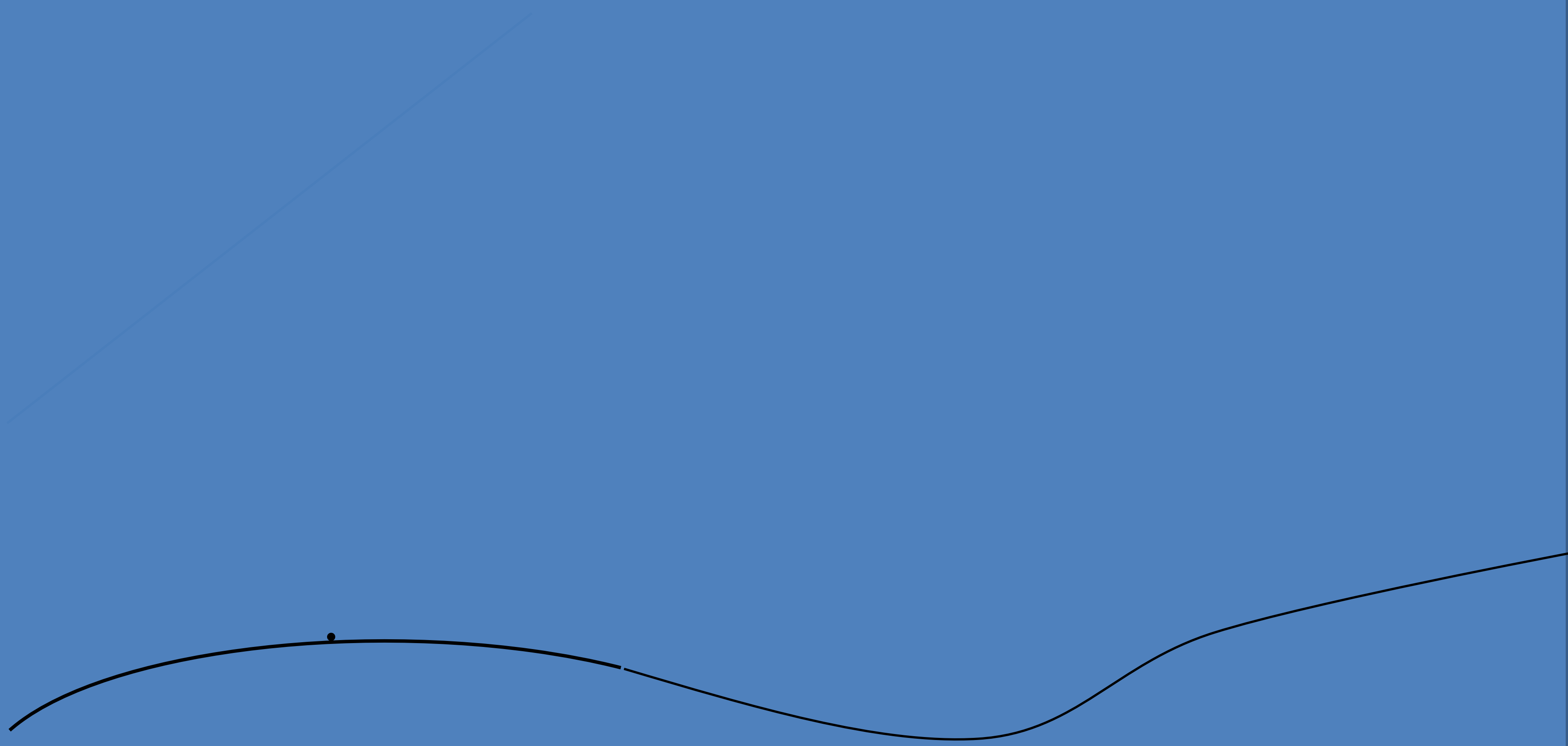


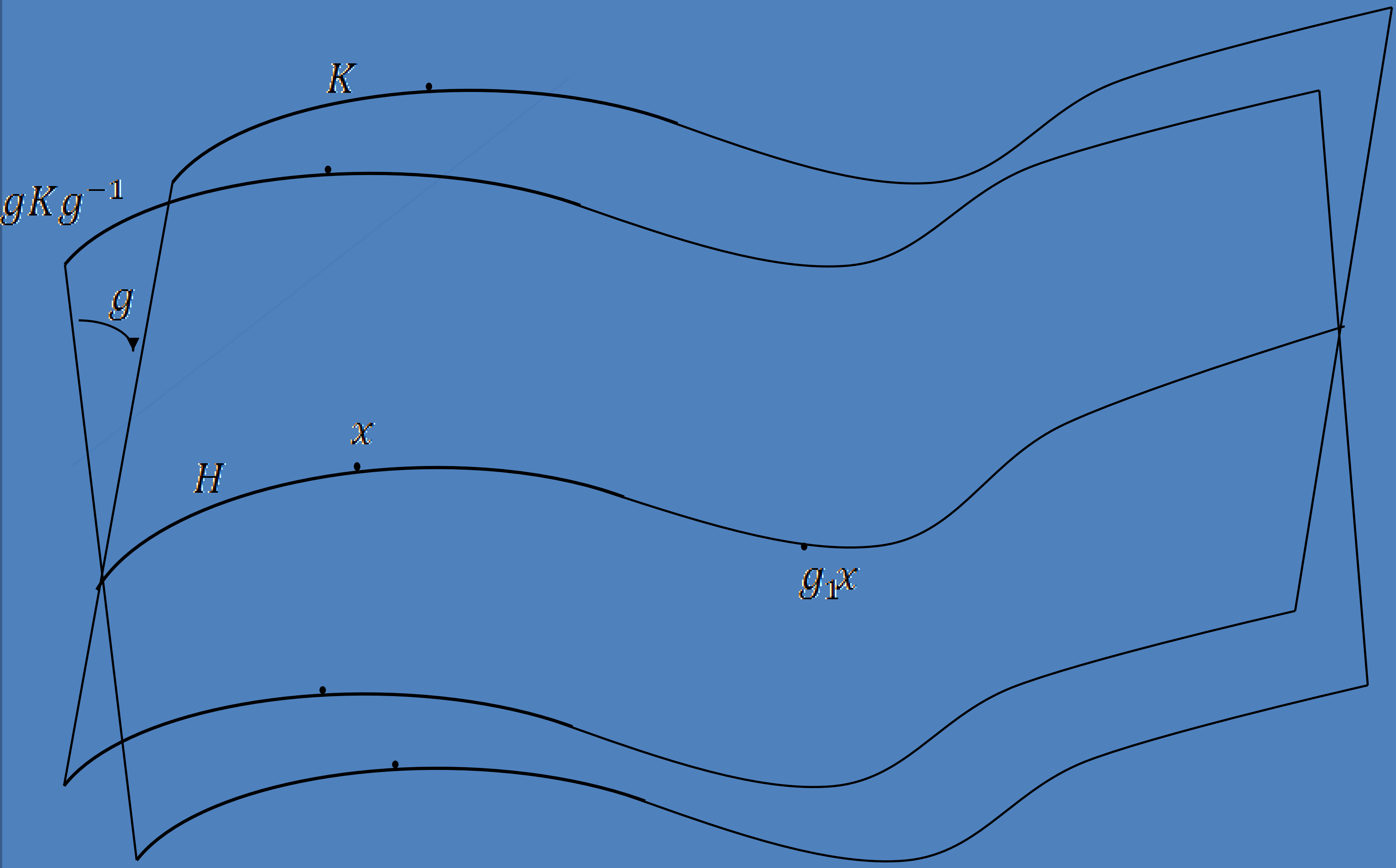
But this is true for any subgroup having non-trivial set of fixed points



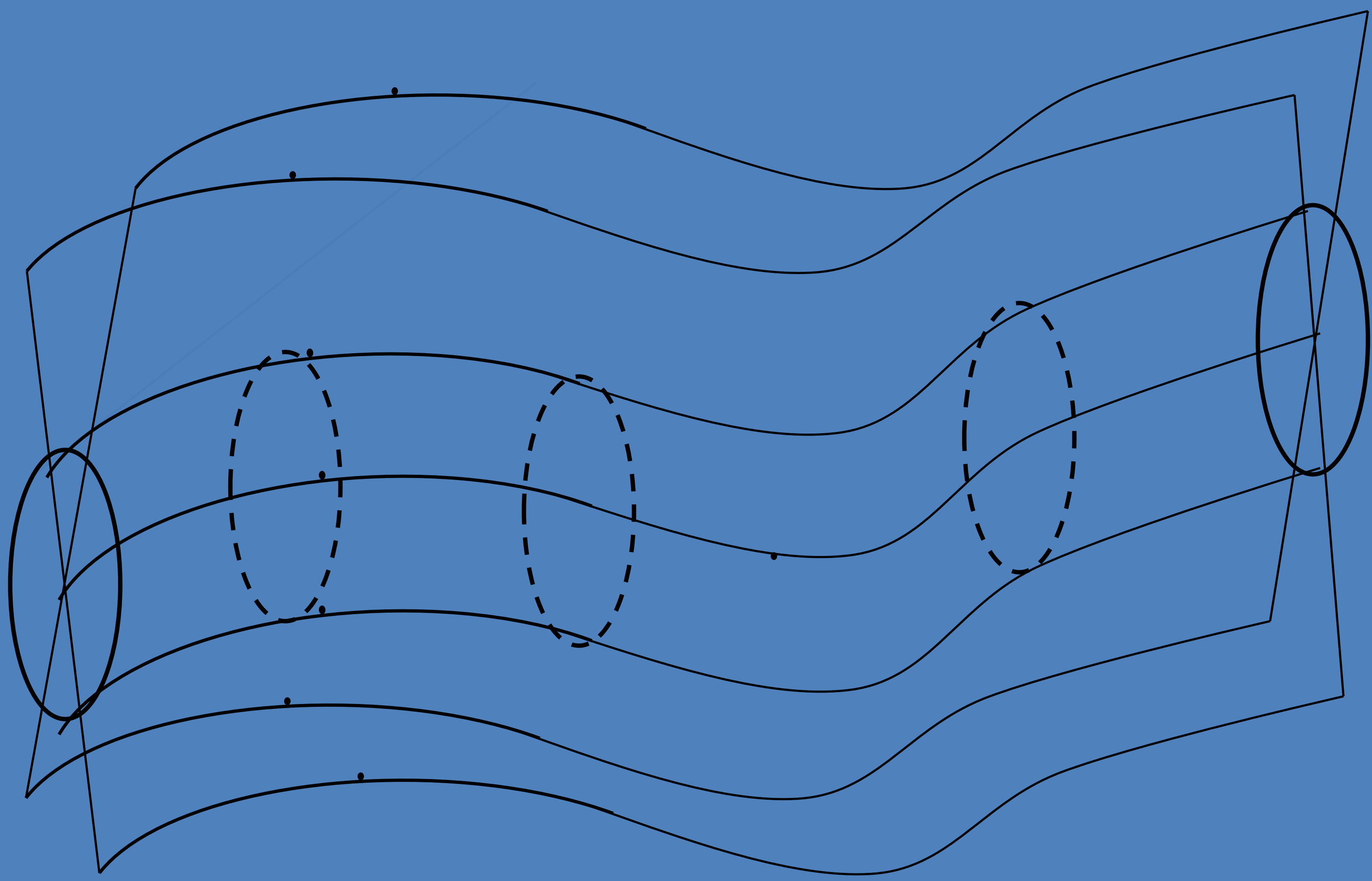
But this is true for any subgroup having non-trivial set of fixed points

But this is true for any subgroup having non-trivial set of fixed points, for example

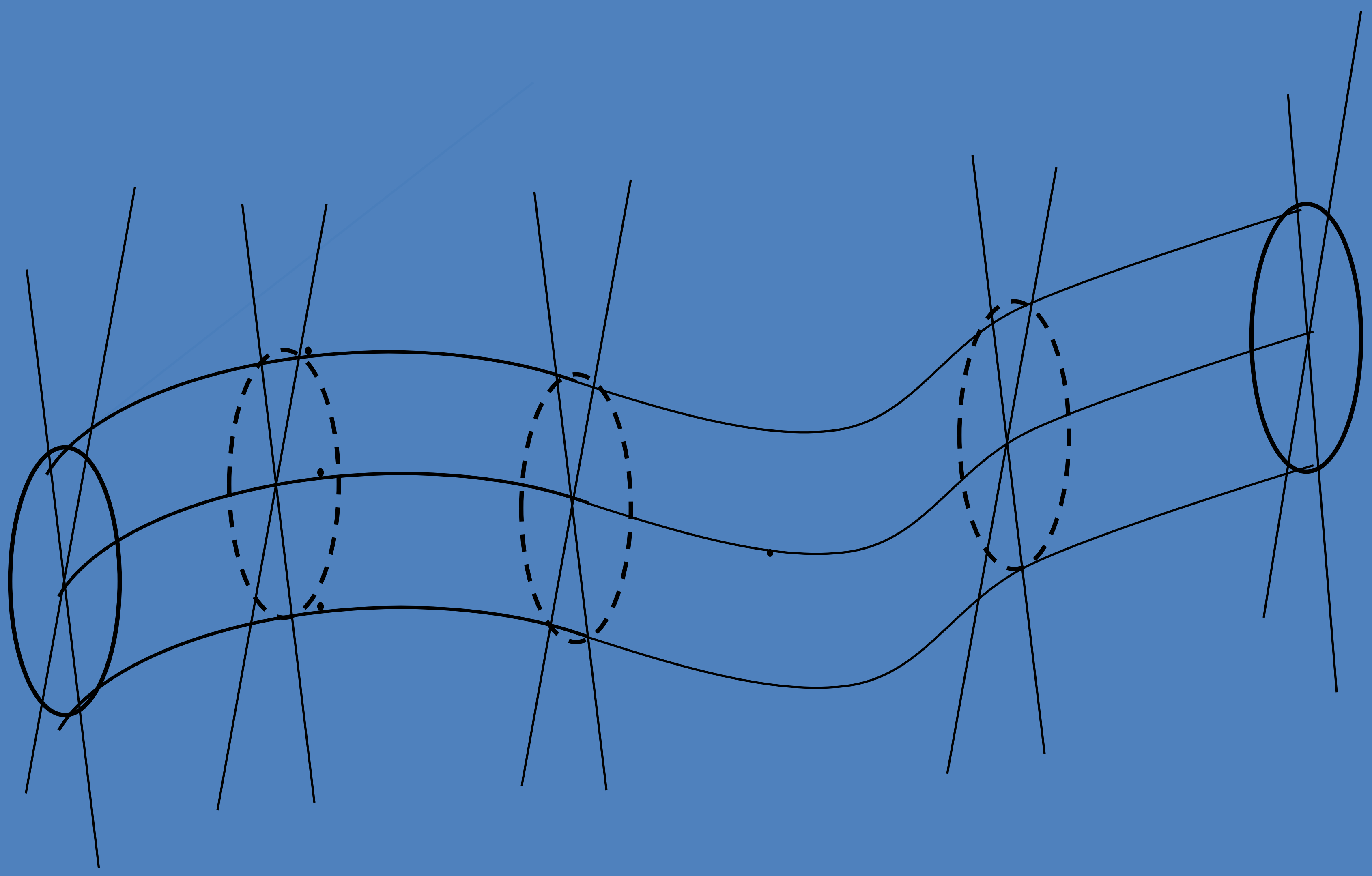
$$K, gKg^{-1} < H$$



We can consider a tubular neighborhood of the fixed-point set

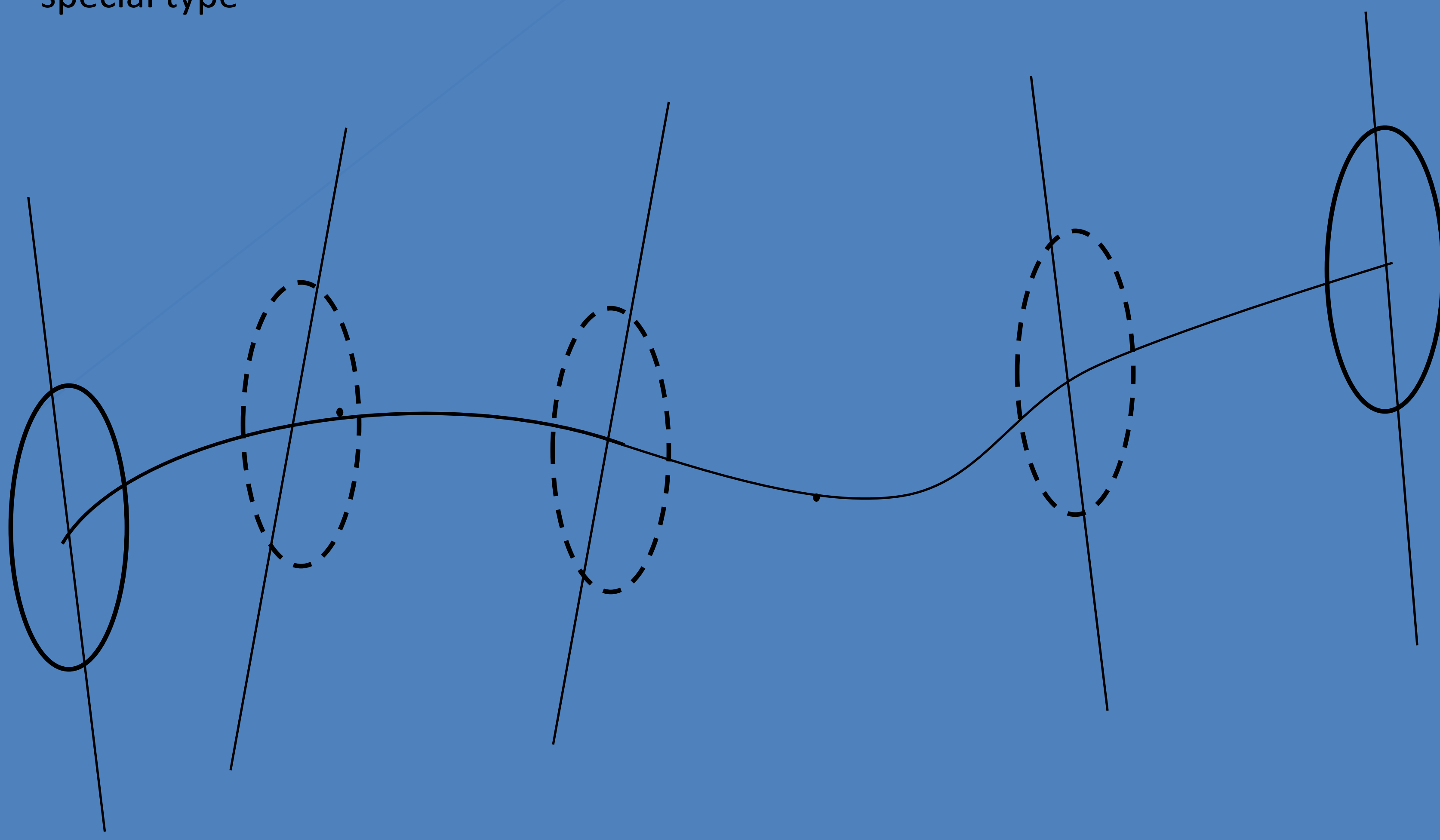


And, if the group H is maximal, then the action of the normalizer N(H) over the fixed –points set M^H is free



And, if the group H is maximal, then the action of the normalizer N(H) over the fixed –points set M^H is free.

Therefore, we can study the fixed-point sets, in terms of equivariant vector bundles of a special type



Equivariant bordisms of cocompact manifolds with proper action of a discrete group

Quitzeh

30.6.2011

Krakow 2011

Some history

In 1945 Bochner[1] showed, as an application of the Haar measure over a compact group, that the action can be linearized on the proximity of a fixed point.

[1] S. Bochner. *Compact groups of differentiable transformations* Ann. of Math **46** (1945), 372–381.

Some history

At the beginning of the 60 Connor and Floyd[2], attracting the modern methods of algebraic topology to solve problems about fixed points, founded the theory of equivariant bordisms. Also, they developed methods for describing the bordisms with **free** action of finite groups in terms of its classifying space, and then applied his construction to calculate the bordisms groups of smooth involutions.

[2] P. Conner, E. Floyd. *Differentiable periodic maps.* Berlin, Springer-Verlag, 1964.

Some history

In 1969, Mishchenko[3] applied this construction to describe the bordisms with action of a cyclic group of odd prime order. He then obtained a long exact sequence for these bordisms in terms of bordisms with proper action and bordisms of manifolds equipped with the structure of its normal (finite-dimensional vector) bundle.

[3] А. С. Мищенко. *Бордизмы с действием группы* $\mathbb{Z}_p$ *и неподвижные точки*. Матем. сборник Т. **80(122)**, № 3(11) (1969), 307–313.

The more recent case

We studied vector bundles

$$\begin{array}{c}\xi\\ \downarrow\\ M\end{array}$$

with quasi-free action of a discrete group G with given stationary normal subgroup H and linear representation $\rho : H \to U(F)$ over the fibers, i.e.

$$\begin{array}{ccc} G \times \xi & \to & \xi \\ \downarrow & & \\ G/H \times M & \to & M, \end{array}$$

The more recent case

And obtained, as a classifying space for these bundles, the classifying space of the group of equivariant automorphisms $B\mathrm{Aut}_G(X_\rho)$ of the **canonical fiber**:

$$X_\rho := G/H \times F$$

with action of the group G

$$\begin{array}{ccc} G \times (G/H \times F) & \stackrel{\phi}{\to} & (G/H \times F) \\ \downarrow & & \downarrow \\ G \times G/H & \stackrel{\mu}{\to} & G/H \end{array}$$

The more recent case

where μ denotes the left action of the group G on its factor G_0, and

$$\phi([g], g_1) : [g] \times F \to [g_1 g] \times F$$

is given by the formula

$$\phi([g], g_1) = \rho(u(g_1 g) u^{-1}(g)),$$

where

$$u : G \to H$$

— is the cocycle defining the group extension

$$1 \to H \to G \to G/H \to 1.$$

The more recent case

The definition of the canonical fiber and of the action of the group G on it depends only on the linear finite-dimensional representation $\rho : H \to U(F)$ of the stationary subgroup H and on the exact sequence (extension) of groups

$$1 \to H \to G \to G/H \to 1.$$

The more recent case

Moreover, as it is known[4] this extension defines an element $u \in H^2(BG, Z(H))$ where $Z(H)$ is the center of the group H. The representation ρ induces then a map

$$\rho_* : H^2(BG, Z(H)) \to H^2(BG, \mathbb{C})$$

because $Z(U(F)) \approx \mathbb{C}$, and it can be proved that the image $\rho_*(u)$ is, actually, the cocycle defining an extension

$$1 \to K \to \mathrm{Aut}_G(X_\rho) \to G/H \to 1.$$

where K is the linear group of matrices commuting with the representation $\rho : H \to U(F)$.

[4] Eilenberg, MacLane *Cohomology Theory in Abstract Groups. II: Group Extensions with a non-Abelian Kernel* Ann. of Math. Second Series, Vol. 48, No. 2 (1947), pp. 326-341

The spectral sequence

Further, we can define the group

$$\Omega_*^{\mathfrak{F},G}$$

of equivariant bordisms of cocompact manifolds with proper action of a discrete group G, where $\mathfrak{F}$ is a family of finite groups of the group G.
$\Omega_*^{\mathfrak{F},G}$ is generated by bordisms classes of manifolds M with proper action $G \times M \to M$ of the discrete group G not having fixed points with respect to the action of finite groups outside the given family $\mathfrak{F}$, also we assume that the corresponding orbit spaces M/G are compact.

The spectral sequence

Here, the analogue of the long exact sequence used for Mishchenko in the case of a cyclic group gives an inductive way to describe group of bordisms of the initial family in terms of bordisms of a smaller family, more precisely, we take out maximal elements of the initial family.

This means that there exists a spectral sequence with respect to this particular filtration in the family of finite groups:

$$\{1\} = \mathfrak{F}_0 \subset \mathfrak{F}_1 \subset \cdots \subset \mathfrak{F}_k = \mathfrak{F}.$$

The first term of this spectral sequence happens to be the group $\Omega_*^{[H]}$ of bordisms equipped with the structure of a vector bundle with quasi-free action of the group G, more exactly

$$E^1_{p,q} \approx \bigoplus_{[H]\in \mathrm{Iso}(\mathfrak{F}_p)/G} \Omega^{[H]}_{p+q}.$$

$\mathrm{Iso}(\mathfrak{F}) \subset \mathfrak{F}$ are the maximal elements.

The spectral sequence

So, we can apply the description of the vector bundles with quasi-free action to calculate this group of bordisms. That is,

$$\Omega_*^{[H]} \approx \bigoplus_{\rho} \Omega_*(B\mathrm{Aut}_{N(H)}(X(\rho)), N(H)/H),$$

The spectral sequence

where $\Omega_*(B\mathrm{Aut}_{N(H)}(X(\rho)), N(H)/H)$ denotes the group of bordisms of the given classifying space.
A class of bordisms in this group is defined by a pair (M, f), where M is some manifold with free action of the factor group $N(H)/H$ and f is a continuous map

$$f : M/\left(N(H)/H\right) \to B\mathrm{Aut}_{N(H)}(X(\rho))$$

References in english lenguage

[1]Alexander S. Mishchenko, Quitzeh Morales Melendez*Description of the vector G-bundles over G-spaces with quasi-free proper action of discrete group G* arXiv:0901.3308v1

[2]Quitzeh Morales Melendez*Description of the vector G-bundles over G-spaces with quasi-free proper action of discrete group II* arXiv:0912.5047v1

[3]M. K. Morales, *Bordisms of manifolds with proper action of a discrete group* Moscow univ. math. bull. (2010) Vol. 65, No. 2, 92-94.