Recent results in Hamilton-Jacobi theory

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Classical Hamilton-Jacobi theory (geometric version)

Let M be the configuration manifold, and T^*M its cotangent bundle equipped with the canonical symplectic form

$$\omega_M = dq^A \wedge dp_A$$

where (q^A) are coordinates in M and (q^A, p_A) are the induced ones in T^*M.

Let $h : T^*M \longrightarrow \mathbb{R}$ a hamiltonian function and X_h the corresponding hamiltonian vector field:

$$i_{X_h} \omega_M = dh$$

The integral curves of X_h, $(q^A(t), p_A(t))$, satisfy the Hamilton equations:

$$\frac{dq^A}{dt} = \frac{\partial h}{\partial p_A}, \quad \frac{dp_A}{dt} = -\frac{\partial h}{\partial q^A}$$

The standard formulation of the Hamilton-Jacobi problem is to find a function $S(t, q^A)$ (called the principal function) such that

$$\frac{\partial S}{\partial t} + h(q^A, \frac{\partial S}{\partial q^A}) = 0. \qquad (1)$$

If we put $S(t, q^A) = W(q^A) - tE$, where E is a constant, then W satisfies

$$h(q^A, \frac{\partial W}{\partial q^A}) = E; \qquad (2)$$

W is called the characteristic function.

Equations (??) and (??) are indistinctly referred as the Hamilton-Jacobi equation.

If $(q^A(t)$ is a solution of

$$\frac{dq^A}{dt} = \frac{\partial h}{\partial p_A}$$

then

$$(q^A(t), \frac{\partial W}{\partial q^A}(q^A(t)))$$

is a solution of the Hamilton equations.

Let λ be a closed 1-form on M, say $d\lambda = 0$; (then, locally $\lambda = dW$)

Hamilton-Jacobi Theorem

The following conditions are equivalent:

(i) If $\sigma : I \to M$ satisfies the equation

$$\frac{dq^A}{dt} = \frac{\partial h}{\partial p_A}$$

then $\lambda \circ \sigma$ is a solution of the Hamilton equations;

(ii) $d(h \circ \lambda) = 0$

Define a vector field on M:

$$X_h^\lambda = T\pi_M \circ X_h \circ \lambda$$

$$\begin{array}{ccc}
T^*M & \xrightarrow{\quad X_h \quad} & T(T^*M) \\
{\scriptstyle \lambda}\uparrow\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle T\pi_M} \\
M & \xrightarrow{\quad X_h^\lambda \quad} & TM
\end{array}$$

The following conditions are equivalent:

(i) If $\sigma : I \to M$ satisfies the equation

$$\frac{dq^A}{dt} = \frac{\partial h}{\partial p_A}$$

then $\lambda \circ \sigma$ is a solution of the Hamilton equations;

(i)' If $\sigma : I \to M$ is an integral curve of X_h^λ, then $\lambda \circ \sigma$ is an integral curve of X_h;

(i)" X_h and X_h^λ are λ-related, i.e.

$$T\lambda(X_h^\lambda) = X_h \circ \lambda$$

Hamilton-Jacobi Theorem

Let λ be a closed 1-form on M. Then the following conditions are equivalent:

(i) X_h^λ **and** X_h **are** λ**-related;**

(ii) $d(h \circ \lambda) = 0$

If

$$\lambda = \lambda_A(q)\, dq^A$$

then the Hamilton-Jacobi equation becomes

$$h(q^A, \lambda_A(q^B)) = const.$$

and we recover the classical formulation when

$$\lambda_A = \frac{\partial W}{\partial q^A}$$

Example: The rolling disk

Consider a disk rolling without sliding on a horizontal plane.

Let (x, y) be the coordinates of the point of contact with the floor, ψ the angle measured from a chosen point of the rim to the point of contact (rotation angle), ϕ is the angle between the tangent to the disk at the point of contact and the x axis (heading angle), and θ is the angle of inclination of the disk.

The configuration manifold is then $Q = \mathbb{R}^2 \times S^1 \times S^1 \times S^1$.

The lagrangian is $L = T - V$ where

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + R^2\dot{\theta}^2 + R^2\dot{\phi}^2 \sin^2\theta) - mR(\dot{\theta}\cos\phi(\dot{x}\sin\phi - \dot{y}\cos\phi) \\ &+ \dot{\phi}\sin\theta(\dot{x}\cos\phi + \dot{y}\sin\phi)) + \frac{1}{2}I_1(\dot{\theta}^2\dot{\phi}^2\cos^2\theta) + \frac{1}{2}I_2(\dot{\psi} + \dot{\phi}\sin\theta)^2 \end{aligned}$$

and

$$V = mgR\cos\theta$$

Here m is the mass of the disk, R is the radius, and I_1 and I_2 are the principal momenta of inertia.

The rolling without sliding condition means that the point of contact has zero velocity and consequently the following constraints have to be fullfilled along the motion

$$\Phi^1 = \dot{x} - (R\cos\phi)\dot{\psi} = 0,\ \Phi^2 = \dot{y} - (R\sin\phi)\dot{\psi} = 0.$$

All the configurations are available, but not all the velocities.

A nonholonomic mechanical system consists of

1. **a lagrangian function** $L = L(q^A, \dot{q}^A)$**,**
2. **subject to nonholonomic constraints** $\Phi^i(q^A, \dot{q}^A) = 0$**.**

> **PROBLEM: How to extend the classical Hamilton-Jacobi theory for nonholonomic mechanical systems?**

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Lagrangian mechanics

Let $L = L(q^A, \dot{q}^A)$ **be a lagrangian function, where** (q^A) **are coordinates in a configuration** n**-manifold** Q**.**

The Hamilton 's principle produces the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^A}\right) - \frac{\partial L}{\partial q^A} = 0, \ 1 \leq A \leq n. \tag{3}$$

A geometric version of Eq. (??) can be obtained as follows.
$L : TQ \longrightarrow \mathbb{R}$. Consider the (1,1)-tensor field S and the Liouville vector field Δ defined on the tangent bundle TQ of Q:

$$S = \frac{\partial}{\partial \dot{q}^A} \otimes dq^A, \ \Delta = \dot{q}^A \frac{\partial}{\partial \dot{q}^A}.$$

We construct the Poincaré-Cartan 1 and 2-forms

$$\alpha_L = S^*(dL), \ \omega_L = -d\alpha_L,$$

S^* denotes the adjoint operator of S.
The energy is given by

$$E_L = \Delta(L) - L,$$

so that we recover the classical expressions

$$\omega_L = dq^A \wedge dp_A, \ E_L = \dot{q}^A p_A - L,$$

$p_A = \dfrac{\partial L}{\partial \dot{q}^A}$ denotes the generalized momenta.

We say that L is regular if the 2-form ω_L is symplectic, which in coordinates turns to be equivalent to the regularity of the Hessian matrix of L with respect to the velocities, say

$$\left(W_{AB} = \frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B} \right)$$

is nonsingular.

In this case, the equation

$$\boxed{i_X\, \omega_L = dE_L \qquad (4)}$$

has a unique solution, $X = \xi_L$, called the Euler-Lagrange vector field; ξ_L is a second order differential equation (SODE) that means that its integral curves are tangent lifts of their projections on Q (these projections are called the solutions of ξ_L). A direct computation shows that the solutions of ξ_L are just the ones of Eqs (??).

If $\flat_L : TTQ \longrightarrow T^TQ$ is the musical isomorphism, $\flat_L(v) = i_v\omega_L$, then we have $\flat_L(\xi_L) = dE_L$.

Legendre transformation

Finally, let us recall that the Legendre transformation $FL : TQ \longrightarrow T^*Q$ is a fibred mapping (that is, $\pi_Q \circ FL = \tau_Q$, where $\tau_Q : TQ \longrightarrow Q$ and $\pi_Q : T^*Q \longrightarrow Q$ denote the canonical projections of the tangent and cotangent bundle of Q, respectively).

In local coordinates, we have

$$FL(q^A, \dot{q}^A) = (q^A, p_A),$$

and then we have that L is regular if and only if FL is a local diffeomorphism.

We will assume that FL is in fact a global diffeomorphism (in other words, L is hyperregular) which is the case when L is a lagrangian of mechanical type, say

$$L = T - V$$

where

- T is the kinetic energy defined by a Riemannian metric on Q,
- $V : Q \longrightarrow \mathbb{R}$ is a potential energy.

Hamiltonian description

The hamiltonian counterpart is developed in the cotangent bundle T^*Q of Q. Denote by $\omega_Q = dq^A \wedge dp_A$ the canonical symplectic form, where (q^A, p_A) are the canonical coordinates on T^*Q.

The Hamiltonian energy is just $H = E_L \circ FL$ and the Hamiltonian vector field is the solution of the symplectic equation

$$i_{X_H}\omega_Q = dH.$$

As we know, the integral curves $(q^A(t), p_A(t))$ of X_H satisfy the Hamilton equations

$$\left.\begin{array}{rcl} \dot{q}^A &=& \frac{\partial H}{\partial p_A} \\ \dot{p}_A &=& -\frac{\partial H}{\partial q^A} \end{array}\right\} \qquad (5)$$

Since $FL^*\omega_Q = \omega_L$ we deduce that ξ_L and X_H are FL-related, and consequently FL transforms the Euler-Lagrange equations (??) into the Hamilton equations (??).

Nonholonomic mechanical systems

A nonholonomic mechanical system is given by a lagrangian function $L = L(q^A, \dot{q}^A)$ subject to a family of constraint functions

$$\Phi^i(q^A, \dot{q}^A) = 0,\ 1 \leq i \leq m \leq n = \dim Q.$$

If $\Phi^i(q^A, \dot{q}^A) = \Phi^i_A(q)\dot{q}^A$ (respectively, $\Phi^i(q^A, \dot{q}^A) = \Phi^i_A(q)\dot{q}^A + b^i(q)$) is linear (respectively, affine) in the velocities the constraints are called linear (respectively, affine). Otherwise, they are called nonlinear.

Invoking the D' Alembert principle for linear and affine constraints (or the Chetaev principle, for nonlinear constraints) we derive the nonholonomic equations of motion

$$\left.\begin{array}{rcl} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^A}\right) - \frac{\partial L}{\partial q^A} & = & \lambda_i \frac{\partial \Phi^i}{\partial \dot{q}^A}\,, 1 \leq A \leq n \\ \Phi^i(q^A, \dot{q}^A) & = & 0. \end{array}\right\} \tag{6}$$

where $\lambda_i = \lambda_i(q^A, \dot{q}^A)$ are Lagrange multipliers to be determined.

In a geometrical setting, L is a function on TQ and the constraints are given by a submanifold M of TQ locally defined by $\Phi^i = 0$.

If the constraints are linear (respectively, affine) then M is the total space of a vector (respectively, affine) subbundle of TQ. For general nonlinear constraints, M is a submanifold satisfying $\tau_Q(M) = Q$ to avoid holonomic constraints. From now on, we will assume nonlinear constraints, since the treatment is the same.

Equations (??) can be equivalently reformulated as follows

$$\left.\begin{array}{lcr} i_X\,\omega_L - dE_L & = & \lambda_i S^*(d\Phi^i) \\ X(\Phi^i) & = & 0 \end{array}\right\} \qquad (7)$$

If we realize that the bundle of 1-forms $S^*((TM)^0)$ is locally generated by the local 1-forms $\{S^*(d\Phi^i)\}$, we can rewrite Eqs (??) as follows

$$\left.\begin{array}{lcr} i_X\,\omega_L - dE_L & \in & S^*((TM)^0) \\ X & \in & TM \end{array}\right\} \tag{8}$$

We assume the *admissibility condition***:**

$$\dim(TM)^0 = \dim S^*((TM)^0)$$

which is equivalent to say that the matrix

$$\left(\frac{\partial \Phi^i}{\partial \dot{q}^A}\right)$$

has maximal rank m.

(For linear constraints the above conditions means that the set of 1-forms $\{\mu^i = \Phi^i_A(q)dq^A\}$ is linearly independent and, indeed, a local cobasis of the distribution M).

We also assume the *compatibility condition*:

$$F^{\perp} \cap TM = \{0\}$$

where F is the distribution on TQ (along M) such that

$$F^0 = S^*((TM)^0)$$

and $F^{\perp}$ denotes the ω_L-complement of F.

Notice that $F^{\perp} = \langle Z^i \rangle$ where $\flat_L(Z^i) = S^*(d\Phi^i)$, therefore $\flat_L(F^{\perp}) = F^0$.

Consider a possible solution of the equation

$$i_X\,\omega_L - dE_L = \lambda_i S^*(d\Phi^i);$$

then $X = \xi_L + \lambda_i Z^i$**. If we impose the condition to the dynamics be tangent to the constraint submanifold we obtain**

$$0 = X(\Phi^j) = \xi_L(\Phi^j) + \lambda_i Z^i(\Phi^j) \tag{9}$$

Denote $C^{ij} = Z^i(\Phi^j)$**. Notice that if the matrix** (C^{ij}) **is regular, then we can compute the Lagrange multipliers solving the linear equation (??) at each point of** M**. In this case we can obtain the nonholonomic dynamics** X_{nh} **which is the unique solution of Eqs. (??).**

A simple calculation gives

$$C^{ij} = \frac{\partial \Phi^i}{\partial \dot{q}^A} W^{AB} \frac{\partial \Phi^j}{\partial \dot{q}^B}$$

where (W^{AB}) **is the inverse matrix of** (W_{AB})**, and shows that if** (W_{AB}) **is definite (positive or negative) then** (C^{ij}) **is inversible.**

As a consequence, if the lagrangian function L **is of mechanical type then the nonholonomic system is admissible and compatible.**

Projections

Assume that the nonholonomic system is compatible and admissible, then we have a direct sum decomposition

$$T_x(TQ) = T_xM \oplus F_x^\perp$$

for all $x \in M$**. In terms of vector bundles we have a Whitney sum decomposition**

$$TTQ_{|M} = TM \oplus F^\perp$$

with two complementary projections $\mathcal{P} : TTQ_{|M} = TM$ **and** $\mathcal{Q} : TTQ_{|M} = F^\perp$ **such that** $X_{nh} = \mathcal{P}(\xi_L)$**.**

To be more precise, the result $X_{nh} = \mathcal{P}(\xi_L)$ **holds if the constraint are homogenous, that is,** Δ **is tangent to the constraint submanifold,** $\Delta_{|M} \in TM$**. This is the case for linear and affine constraints.**

Assuming the regularity of the Lagrangian, we have that the Lagrangian and Hamiltonian formulations are locally equivalent. If we suppose, in addition, that the Lagrangian L is hyperregular, then the Legendre transformation $FL : TQ \to T^*Q, (q^A, \dot{q}^A) \mapsto (q^A, p_A = \partial L/\partial \dot{q}^A)$, is a global diffeomorphism. The constraint functions on T^*Q become $\Psi^i = \Phi^i \circ FL^{-1}$, i.e.

$$\Psi^i(q^A, p_A) = \Phi^i(q^A, \frac{\partial H}{\partial p_A}) ,$$

where the Hamiltonian $H : T^*Q \to \mathbb{R}$ is defined by $H = E_L \circ FL^{-1}$. Since locally $FL^{-1}(q^A, p_A) = (q^A, \frac{\partial H}{\partial p_A})$, then

$$H = p_A \dot{q}^A - L(q^A, \dot{q}^A) ,$$

where $\dot{q}^A$ is expressed in terms of q^A and p_A using FL^{-1}.

The equations of motion for the nonholonomic system on T^*Q can now be written as follows

$$\left.\begin{array}{rcl} \dot{q}^A &=& \dfrac{\partial H}{\partial p_A} \\ \dot{p}_A &=& -\dfrac{\partial H}{\partial q^A} - \bar{\lambda}_i \dfrac{\partial \Psi^i}{\partial p_B}\mathcal{H}_{BA} \end{array}\right\} \qquad (10)$$

together with the constraint equations

$$\Psi^i(q,p) = 0$$

where $\mathcal{H}_{AB}$ **are the components of the inverse of the matrix** $(\mathcal{H}^{AB}) = (\partial^2 H/\partial p_A \partial p_B)$**. Note that**

$$\left(\frac{\partial \Psi^i}{\partial p_B}\mathcal{H}_{BA}\right)(q,p) = \left(\frac{\partial \Phi^i}{\partial \dot{q}^A} \circ FL^{-1}\right)(q,p).$$

The symplectic 2-form ω_L is related, via the Legendre map, with the canonical symplectic form ω_Q on T^*Q. Let $\bar{M}$ denote the image of the constraint submanifold M under the Legendre transformation, and let $\bar{F}$ be the distribution on T^*Q along $\bar{M}$, whose annihilator is given by

$$\bar{F}^0 = FL_*(S^*((T\bar{M})^0))\,.$$

Observe that $\bar{F}^0$ is locally generated by the m independent 1-forms

$$\bar{\mu}^i = \frac{\partial \Psi^i}{\partial p_A}\mathcal{H}_{AB}dq^B\;,\;\; 1 \leq i \leq m\,.$$

The nonholonomic Hamilton equations for the nonholonomic system can be then rewritten in intrinsic form as

$$\left.\begin{array}{rcl} (i_X\omega_Q - dH)_{|\bar{M}} & \in & \bar{F}^0 \\ X_{|\bar{M}} & \in & T\bar{M} \end{array}\right\} \qquad (11)$$

The compatibility condition is now written as $\bar{F}^{\perp} \cap T\bar{M} = \{0\}$**, where "** $\perp$ **" denotes the symplectic complement with respect to** ω_Q**. Equivalently, the matrix**

$$(\bar{C}^{ij}) = \left(\frac{\partial \Psi^i}{\partial p_A} \mathcal{H}_{AB} \frac{\partial \Psi^j}{\partial p_B} \right) \tag{12}$$

is regular. On the Lagrangian side, the compatibility condition is locally written as

$$\det(\bar{C}^{ij}) = \det \left(\frac{\partial \phi^i}{\partial \dot{q}^A} W^{AB} \frac{\partial \phi^j}{\partial \dot{q}^B} \right) \neq 0 \ , \tag{13}$$

where W^{AB} **are the entries of the Hessian matrix** $\left(\dfrac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B} \right)_{1 \leq A,B \leq n}$**.**

The compatibility condition is not too restrictive, since it is trivially verified by the usual systems of mechanical type (lagrangian = kinetic energy - potential energy), where the $\mathcal{H}_{AB}$ represent the components of a positive definite Riemannian metric. The compatibility condition guarantees the existence of a unique solution of the constrained equations of motion (??) which, henceforth, will be denoted by $\bar{X}_{nh}$ on the Hamiltonian side and X_{nh} on the Lagrangian side. Moreover, if X_H is the Hamiltonian vector field of H ($i_{X_H}\omega_Q = dH$) then

$$\bar{\lambda}_i = \bar{\mathcal{C}}_{ij} X_H(\Psi^j) \ . \tag{14}$$

Hamilton-Jacobi theory for nonholonomic mechanical systems

Let $L : TQ \longrightarrow \mathbb{R}$ be a lagrangian function subject to nonholonomic constraints given by a submanifold M of TQ. We assume the admissibility and compatibility conditions, and consider the hamiltonian counterpart given by a Hamiltonian function $H : T^*Q \longrightarrow \mathbb{R}$ and a constraint submanifold $\bar{M} = FL(M)$ as in the precedent sections. X_{nh} and $\bar{X}_{nh}$ will denote the corresponding nonholonomic dynamics.

Let γ be a closed 1-form on Q such that $\gamma(Q) \subset \bar{M}$. Then the following conditions are equivalent:

(i) for every curve $\sigma : \mathbb{R} \longrightarrow Q$ such that

$$\dot{\sigma}(t) = T\pi_Q(X_H(\gamma(\sigma(t)))) \tag{15}$$

for all t, then $\gamma \circ \sigma$ is an integral curve of $\bar{X}_{nh}$.

(ii) $\pi_Q^*(d(H \circ \gamma)) \in \bar{F}^0$.

Let $L : TQ \longrightarrow \mathbb{R}$ a lagrangian subject to linear constraints given by a distribution M on Q. Denote by $\bar{M} \subset T^*Q$ the image of $M \subset TQ$ by the Legendre transformation, and by h the corresponding hamiltonian function on T^*Q. In that case, we have proved the following result:

Hamilton-Jacobi Theorem

Let λ be a 1-form on Q taking values into $\bar{M}$ and satisfying $d\lambda \in \mathcal{I}(M^o)$. Then the following conditions are equivalent:

(i) $\bar{X}_{nh}^{\lambda}$ **and** $\bar{X}_{nh}$ **are** λ**-related;**

(ii) $d(h \circ \lambda) \in M^o$

Here, X_{nh} is the nonholonomic dynamics.

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Basic tools in Classical Hamilton-Jacobi theory

$TM \xrightarrow{\tau_{TM}} M$ $\rightsquigarrow$ **vector bundle over a manifold** M

The canonical symplectic 2-form ω_M **in** $T^*M \simeq$ **The canonical Poisson** 2**-vector** Λ_{T^*M} **on** T^*M $\rightsquigarrow$ **a linear bivector on the dual of the vector bundle.**

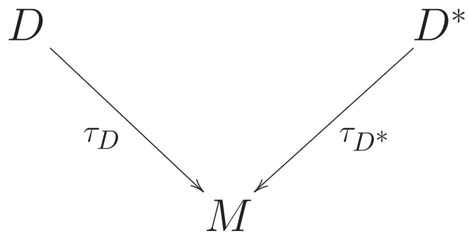
A hamiltonian function $h : T^*M \longrightarrow \mathbb{R}$ $\rightsquigarrow$ **A function** h **defined on the dual of the vector bundle**

A section $\lambda : M \longrightarrow T^*M$ **such that** $d\lambda = 0$ $\rightsquigarrow$ **A section of the dual of the vector bundle which is closed with respect to the "induced differential".**

Geometric Hamilton-Jacobi Theory

Ingredients:

- $\tau_D : D \longrightarrow M$ **a** vector bundle**, and** $\tau_{D^*} : D^* \longrightarrow M$ **its dual vector bundle.**



- **A** linear bivector[1] Λ_{D^*} **on** D^* **(not Jacobi identity is required). We denote by** $\{\ ,\ \}_{D^*}$ **the corresponding almost-Poisson bracket.**

- $h : D^* \longrightarrow \mathbb{R}$ **a** hamiltonian function**.**

[1] linear means that the bracket of two linear functions is a linear function

Λ_{D^*} **is linear**

$$\Downarrow$$

Proposition 1

We have that:

(a) $\xi_1, \xi_2 \in \Gamma(\tau_D) \Rrightarrow \{\widehat{\xi_1}, \widehat{\xi_2}\}_{D^*}$ **is a linear function on** D^***,**

(b) $\xi \in \Gamma(\tau_D)$**,** $f \in C^\infty(M) \Rrightarrow \{\widehat{\xi}, f \circ \tau_{D^*}\}_{D^*}$ **is a basic function with respect to** τ_{D^*}**,**

(c) $f, g \in C^\infty(M) \Rrightarrow \{f \circ \tau_{D^*}, g \circ \tau_{D^*}\}_{D^*} = 0$

$$\Downarrow$$

Given local coordinates (x^μ) in the base manifold M and a local basis of sections of D, $\{e_\alpha\}$, we induce local coordinates (x^μ, y_α) on D^* and the bivector Λ_{D^*} is written as

$$\Lambda_{E^*} = \rho^\mu_\alpha \frac{\partial}{\partial y_\alpha} \wedge \frac{\partial}{\partial x^\mu} + \frac{1}{2} C^\gamma_{\alpha\beta} y_\gamma \frac{\partial}{\partial y_\alpha} \wedge \frac{\partial}{\partial y_\beta}$$

The corresponding Hamiltonian vector field is

$$X_h = \sharp_{\Lambda_{D^*}}(dh)$$

or, in coordinates,

$$X_h = \rho^\mu_\alpha \frac{\partial h}{\partial y_\alpha} \frac{\partial}{\partial x^\mu} - \left(\rho^\mu_\alpha \frac{\partial h}{\partial x^\mu} + C^\gamma_{\alpha\beta} y_\gamma \frac{\partial h}{\partial y_\beta} \right) \frac{\partial}{\partial y_\alpha}$$

Thus, the Hamilton equations are

$$\frac{dx^\mu}{dt} = \rho^\mu_\alpha \frac{\partial h}{\partial y_\alpha}, \quad \frac{dy_\alpha}{dt} = - \left(\rho^\mu_\alpha \frac{\partial h}{\partial x^\mu} + C^\gamma_{\alpha\beta} y_\gamma \frac{\partial h}{\partial y_\beta} \right)$$

The skew-symmetric algebroid structure on $\tau_D : D \longrightarrow M$

The linear bivector Λ_{D^*} induces the following structure on D:

- **a skew-symmetric bracket on the space $\Gamma(\tau_D)$**

$$\begin{array}{rccc} [\ ,\ ]_D : & \Gamma(\tau_D) \times \Gamma(\tau_D) & \longrightarrow & \Gamma(\tau_D) \\ & (\xi_1, \xi_2) & \longmapsto & [\xi_1, \xi_2]_D \end{array}$$

where $\widehat{[\xi_1, \xi_2]_D} = \{\widehat{\xi_1}, \widehat{\xi_2}\}_{D^*}$ $([e_\alpha, e_\beta]_D = C^\gamma_{\alpha\beta} e_\gamma)$**.**

- **an anchor map $\rho_D : \Gamma(\tau_D) \longrightarrow \mathfrak{X}(M)$**

$$f \in C^\infty(M), \xi \in \Gamma(D) \Rrightarrow \rho_D(\xi)(f) \circ \tau_{D^*} = \{\widehat{\xi}, f \circ \tau_{D^*}\}_{D^*}$$

(in coordinates, $\rho_D(e_\alpha) = \rho^\mu_\alpha \frac{\partial}{\partial x^\mu}$).

Properties

a) $[\ ,\ ]_D$ **is antisymmetric**

b) $[\xi_1, f\xi_2]_D = f[\xi_1, \xi_2]_D + \rho_D(\xi_1)(f)\xi_2$

In general, $[\ ,\ ]_D$ **does not satisfy the Jacobi identity. In the case when it satisfies the Jacobi identity we say that** $(D, [\ ,\ ]_D, \rho_D)$ **is a Lie algebroid.**

The almost differential $d^D : \Gamma(\Lambda^k D^*) \longrightarrow \Gamma(\Lambda^{k+1} D^*)$

Given $\Omega \in \Gamma(\Lambda^k D^*)$ **then** $d^D\Omega \in \Gamma(\Lambda^{k+1} D^*)$ **and**

$$
\begin{aligned}
d^D\Omega(\xi_0, \xi_1, \ldots, \xi_k) \;=\; & \sum_{i=0}^{k} (-1)^i \rho_D(\xi_i)(\Omega(\xi_0, \ldots, \widehat{\xi_i}, \ldots, \xi_k)) \\
& + \sum_{i<j} \Omega([\xi_i, \xi_j]_D, \xi_0, \ldots, \widehat{\xi_i}, \ldots, \widehat{\xi_j}, \ldots, \xi_k)
\end{aligned}
$$

where $\xi_0, \xi_1, \ldots, \xi_k \in \Gamma(\tau_D)$

From the definition, we deduce that

$$
\begin{aligned}
(1)\; (d^D f)(\xi) \;&=\; \rho_D(\xi)(f), \quad f \in C^\infty(M), \quad \xi \in \Gamma(\tau_D) \\
(2)\; d^D\sigma(\xi_1, \xi_2) \;&=\; \rho_D(\xi_1)(\sigma(\xi_2)) - \rho_D(\xi_2)(\sigma(\xi_1)) - \sigma[\xi_1, \xi_2]_D, \\
& \quad\; \sigma \in \Gamma(\tau_{D^*}), \quad \xi_1, \xi_2 \in \Gamma(\tau_D) \\
(3)\; d^D(\Omega \wedge \Omega') \;&=\; d^D\Omega \wedge \Omega' + (-1)^k \Omega \wedge d^D\Omega', \quad \Omega \in \Gamma(\Lambda^k D^*), \Omega' \in \Gamma(\Lambda^{k'} D^*)
\end{aligned}
$$

In general $\boxed{(d^D)^2 \neq 0}$**.**

A linear bivector Λ_{D^*} on D^*

$\Downarrow$

A skew-symmetric algebroid structure $([\ ,\ ]_D, \rho_D)$ on D

$\Downarrow$

An almost differential $d^D : \Gamma(\Lambda^k D^*) \longrightarrow \Gamma(\Lambda^{k+1} D^*)$ satisfying (1) and (2)

The inverse process also works

An almost differential $d^D \colon \Gamma(\Lambda^k D^*) \to \Gamma(\Lambda^{k+1} D^*)$ **satisfying (1) and (2)**

$$\Downarrow$$

A skew-symmetric algebroid structure $([\ ,\ ]_D, \rho_D)$ **on** D

$$\rho_D(\xi)(f) = d^D(f)(\xi),$$
$$\omega([\xi, \xi']_D) = -(d^D\omega)(\xi, \xi') + d^D(\omega(\xi'))(\xi) - d^D(\omega(\xi))(\xi')$$
$$\xi, \xi' \in \Gamma(\tau_D), \quad f \in C^\infty(D), \quad \omega \in \Gamma(\tau_{D^*})$$

A skew-symmetric algebroid structure $([\ ,\ ]_D, \rho_D)$ **on** D

$$\Downarrow$$

A linear bivector Λ_{D^*} **on** D^* **with almost Poisson bracket** $\{\ ,\ \}_{D^*}$

$$\{\widehat{\xi}, \widehat{\xi'}\}_{D^*} = \widehat{[\xi, \xi']_D}, \quad \{\widehat{\xi}, f \circ \tau_{D^*}\}_{D^*} = \rho_D(\xi)(f) \circ \tau_{D^*},$$

$$\{f \circ \tau_{D^*}, f' \circ \tau_{D^*}\}_{D^*} = 0$$

$$f, f' \in C^\infty(M), \quad \xi, \xi' \in \Gamma(\tau_D)$$

In conclusion

A linear bivector Λ_{D^*} **on** D^*

$$\Updownarrow$$

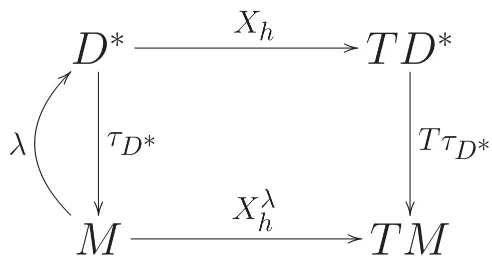
A skew-symmetric Lie algebroid structure $([\ ,\ ]_D, \rho_D)$**on** D

$$\Updownarrow$$

An almost differential $d^D\colon\Gamma(\Lambda^k D^*)\to\Gamma(\Lambda^{k+1} D^*)$**satisfying (1) and (2)**

Hamilton-Jacobi Theorem

Let Λ_{D^*} **be a linear bivector on** D **and** $\lambda : M \longrightarrow D^*$ **be a section of** $\tau_{D^*} : D^* \longrightarrow M$



We define $X_h^\lambda = T\tau_{D^*} \circ X_h \circ \lambda$

It is easy to show that $X_h^\lambda(x) \in \rho_D(D_x)$**,** $\forall x \in M$
Indeed, look the local expressions

$$X_h^\lambda = \rho_\alpha^\mu \frac{\partial h}{\partial y_\alpha} \frac{\partial}{\partial x^\mu} = \rho(\frac{\partial h}{\partial y_\alpha} e_\alpha)$$

Hamilton-Jacobi Theorem

Assume that $d^D\lambda = 0$**.**

(i) $\sigma : I \to M$ **integral curve of** $X_h^\lambda \Rightarrow \lambda \circ \sigma$ **integral curve of** X_h

$$\Updownarrow$$

$$(ii) \quad d^D(h \circ \lambda) = 0$$

For the proof, we will need the following preliminary results (Propositions 2 and 3).

Proposition 2

Let $\lambda : M \longrightarrow D^*$ be a section of τ_{D^*}. Then, λ is a 1-cocycle with respect to d^D (i.e. $d^D\lambda = 0$)

$$\Updownarrow$$

for all $x \in M$ the subspace

$$\mathcal{L}_{\lambda,D}(x) = (T_x\lambda)(\rho_D(D_x)) \subseteq T_{\lambda(x)}D^*$$

is Lagrangian with respect to Λ_{D^*}, that is,

$$\sharp_{\Lambda_{D^*}}(\mathcal{L}_{\lambda,D})^o = \mathcal{L}_{\lambda,D}$$

Remark: Proposition 2 is the generalization of the well-known result for the particular case $D = TM$ and $\Lambda_{D^*} = \Lambda_{T^*M}$:

> **"Let λ be a 1-form on M and $\mathcal{L}_{\lambda,TM} \subseteq T(T^*M)$ the tangent bundle of the image $i(\lambda)$ of λ. Then, λ is closed if and only if $i(\lambda)$ is a Lagrangian submanifold of T^*M."**

Proposition 3

Let $\lambda : M \longrightarrow D^*$ **be a section of** τ_{D^*} **such that** $d^D\lambda = 0$**. Then**

$$(\ker \sharp_{\Lambda_{D^*}})_{\lambda(x)} \subseteq (\mathcal{L}_{\lambda,D})^o, \text{ for all } x \in M$$

Remark: In the particular case when $D = TM$ **this Proposition is trivial since**

$$\ker \sharp_{\Lambda_{T^*M}} = \{0\}$$

((T^*M, ω_M) **is a symplectic manifold).**

Remember that

$$\sharp_{\Lambda_{D^*}} : T^*_{\lambda(x)}D^* \longrightarrow T_{\lambda(x)}D^*$$

Proof of the Theorem

Let $\lambda : M \longrightarrow D^*$ **be a section such that** $d^D\lambda = 0$**.**

(i) ⇒ (ii)

We assume that the integral curves of X_h^λ **and** X_h **are** λ**-related, that is,** X_h^λ **and** X_h **are** λ**-related.**

Moreover, we know that $X_h^\lambda(x) \in \rho_D(D_x)$, $\forall x \in M$**.**

Therefore, $X_h(\lambda(x)) \in (T_x\lambda)(\rho_D(D_x)) = \mathcal{L}_{\lambda,D}(x)$**, for all** $x \in M$**.**

From Proposition 1 ($\mathcal{L}_{\lambda,D}$ **is lagrangian) we deduce that**

$$X_h(\lambda(x)) = \sharp_{\Lambda_{D^*}}(\eta_{\lambda(x)}), \text{ for some } \eta_{\lambda(x)} \in (\mathcal{L}_{\lambda,D})^o$$

Moreover from the definition of hamiltonian vector field

$$X_h(\lambda(x)) = \sharp_{\Lambda_{D^*}}(dh(\lambda(x))$$

Thus,

$$\eta_{\lambda(x)} - dh(\lambda(x)) \in \ker \sharp_{\Lambda_{D^*}}(\lambda(x)) \subseteq \mathcal{L}_{\lambda,D}(x)^o \text{ (by Proposition 2)}$$

Then, $dh(\lambda(x)) \in \mathcal{L}_{\lambda,D}(x)^o, \forall x \in M$**.**

Finally, if $a_x \in D_x$**, then**

$$\begin{aligned} d^D(h \circ \lambda)(x)(a_x) &= \rho_D(a_x)(h \circ \lambda) = (T_x\lambda)(\rho_D(a_x))(h) \\ &= dh(\lambda(x))(T_x\lambda)(\rho_D(a_x)) = 0 \end{aligned}$$

(ii) $\Rightarrow$ (i)

The condition $d^D(h \circ \lambda) = 0$ implies that

$$dh(\lambda(x)) \in \mathcal{L}_{\lambda,D}(x)^o, \forall x \in M$$

Then

$$\begin{aligned} X_H(\lambda(x)) &= \sharp_{\Lambda_{D^*}}(dh)(\lambda(x)) \in \sharp_{\Lambda_{D^*}}(\mathcal{L}_{\lambda,D}(x)^o) = \mathcal{L}_{\lambda,D} \text{ (Proposition 1)} \\ &= T_x\lambda(\rho_D(D_x)) \end{aligned}$$

Therefore X_h^λ and X_h are λ-related and we conclude (i). □

Local expression of the Hamilton-Jacobi equations

Take local coordinates (x^μ) in the base manifold M, a local basis of sections of D, $\{e_\alpha\}$, and induced coordinates (x^μ, y_α) on D^*. Then if

$$\lambda : (x^\mu) \longrightarrow (x^\mu, \lambda_\alpha(x^\mu)) \equiv (x, \lambda(x))$$

we have

$$d^D(h \circ \lambda) = 0$$

is locally written as

$$\begin{aligned} 0 &= d^D(h \circ \lambda)(e_\alpha)_x \\ &= \rho_D(x)(e_\alpha(x))(h \circ \lambda) \\ &= \rho^\mu_\alpha(x) \frac{\partial}{\partial x^\mu}(h \circ \lambda)_x \\ &= \rho^\mu_\alpha(x) \left[\frac{\partial h}{\partial x^\mu}(x, \lambda(x)) + \frac{\partial h}{\partial y_\beta}(x, \lambda(x)) \frac{\partial \lambda_\beta}{\partial x^\mu}(x) \right], \quad \forall \alpha \end{aligned}$$

The Hamilton-Jacobi Equations

$$\rho^\mu_\alpha(x) \left[\frac{\partial h}{\partial x^\mu}(x, \lambda(x)) + \frac{\partial h}{\partial y_\beta}(x, \lambda(x)) \frac{\partial \lambda_\beta}{\partial x^\mu}(x) \right] = 0$$

Application: Mechanical systems with nonholonomic constraints

Let $\mathcal{G} : E \times_M E \to \mathbb{R}$ **be a bundle metric on a Lie algebroid** $(E, [\cdot\,,\cdot], \rho)$

The class of systems that were considered is that of ***mechanical systems with nonholonomic constraints*** **determined by:**

- **The Lagrangian function** L**:**

$$L(a) = \frac{1}{2}\mathcal{G}(a, a) - V(\tau(a)), \quad a \in E,$$

  with V **a function on** M

- **The nonholonomic constraints determined by a subbundle** D **of** E

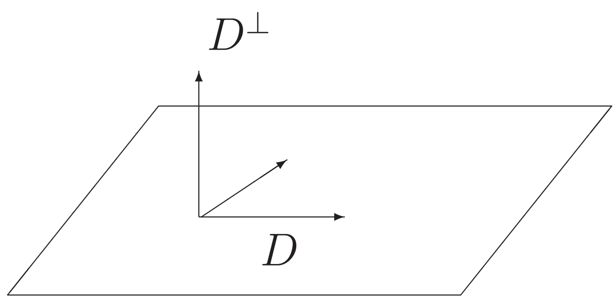
Consider the orthogonal decomposition $E = D \oplus D^\perp$, and the associated orthogonal projectors

$$P \;:\; E \longrightarrow D$$
$$Q \;:\; E \longrightarrow D^\perp$$

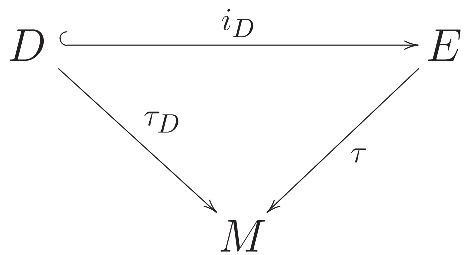
Take local coordinates (x^μ) in the base manifold M and a local basis of sections of E (moving basis), $\{e_\alpha\}$, adapted to the nonholonomic problem (L, D), in the sense that

(i) $\{e_\alpha\}$ is an orthonormal basis with respect to $\mathcal{G}$ (that is $\mathcal{G}(e_\alpha, e_\beta) = \delta_{\alpha\beta}$)

(ii) $\{e_\alpha\} = \{e_a, e_A\}$ where $D = \mathrm{span}\{e_a\}$, $D^\perp = \mathrm{span}\{e_A\}$.



Denoting by $(x^\mu, y^\alpha) = (x^\mu, y^a, y^A)$ the induced coordinates on E, the constraint equations determining D just read $\boxed{y^A = 0.}$ Therefore we choose (x^μ, y^a) as a set of coordinates on D



In these coordinates we have the inclusion

$$\begin{array}{rccc} i_D : & D & \longrightarrow & E \\ & (x^\mu, y^a) & \longmapsto & (x^\mu, y^a, 0) \end{array}$$

and the dual map

$$\begin{array}{rccc} i_D^* : & E^* & \longrightarrow & D^* \\ & (x^\mu, y_a, y_A) & \longmapsto & (x^\mu, y_a) \end{array}$$

where (x^μ, y_α) are the induced coordinates on E^* by the dual basis of $\{e_\alpha\}$.

Moreover, from the orthogonal decomposition we have that

$$\begin{array}{rrcl} P: & E & \longrightarrow & D \\ & (x^\mu, y^a, y^\alpha) & \longmapsto & (x^\mu, y^a) \end{array}$$

and its dual map

$$\begin{array}{rrcl} P^*: & D^* & \longrightarrow & E^* \\ & (x^\mu, y_a) & \longmapsto & (x^\mu, y_a, 0) \end{array}$$

In these coordinates, the nonholonomic system is given by

i) The Lagrangian $L(x^\mu, y^\alpha) = \frac{1}{2}\sum_\alpha (y^\alpha)^2 - V(x^\mu)$**,**

ii) The nonholonomic constraints $y^A = 0$**.**

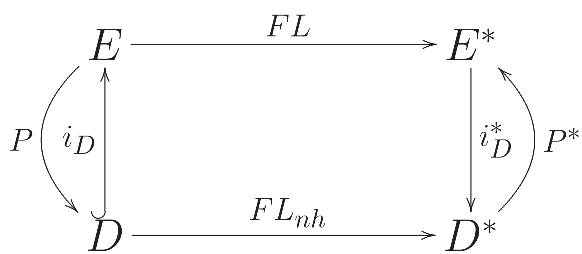
In this case, the Legendre transformation associated with L is the isomorphism $FL : E \longrightarrow E^*$ induced by the metric $\mathcal{G}$. Therefore, locally, the Legendre transformation is

$$\begin{array}{rccc} FL: & E & \longrightarrow & E^* \\ & (x^\mu, y^\alpha) & \longmapsto & (x^\mu, y_\alpha = y^\alpha) \end{array}$$

and we can define the ***nonholonomic Legendre transformation*** $FL_{nh} = i_D^* \circ FL \circ i_D : D \longrightarrow D^*$

$$\begin{array}{rccc} FL_{nh}: & D & \longrightarrow & D^* \\ & (x^\mu, y^a) & \longmapsto & (x^\mu, y_a = y^a) \end{array}$$

Summarizing, we have the following diagram



$(E, [\ ,\ ], \rho)$ **is a Lie algebroid**

$$\Downarrow$$

Λ_{E^*} **is a linear Poisson structure on** E^*

If f_1 **and** f_2 **are functions on** M**, and** ξ_1 **and** ξ_2 **are sections of** E**, then:**

$$\{f_1 \circ \tau_{E^*}, g_1 \circ \tau_{E^*}\}_{E^*} = 0, \quad \{\widehat{\xi_1}, f_1 \circ \tau_{E^*},\}_{E^*} = (\rho(\xi_1))\, f_1 \circ \tau_{E^*}, \quad \{\widehat{\xi_1}, \widehat{\xi_2}\}_{E^*} = \widehat{[\xi_1, \xi_2]}$$

In the induced coordinates (x^μ, y_α)**, the Poisson bracket relations on** E^* **are**

$$\{x^\mu, x^\eta\}_{E^*} = 0, \qquad \{y_\alpha, x^\mu,\}_{E^*} = \rho^\mu_\alpha, \qquad \{y_\alpha, y_\beta\}_{E^*} = C^\gamma_{\alpha\beta} y_\gamma$$

In other words

$$\Lambda_{E^*} = \rho^\mu_\alpha \frac{\partial}{\partial y_\alpha} \wedge \frac{\partial}{\partial x^\mu} + \frac{1}{2} C^\gamma_{\alpha\beta} y_\gamma \frac{\partial}{\partial y_\alpha} \wedge \frac{\partial}{\partial y_\beta}$$

The *nonholonomic bracket* on D^*, $\{\ ,\ \}_{nh,D^*}$, is defined by

$$\boxed{\{F,G\}_{nh,D^*} = \{F \circ i_D^*, G \circ i_D^*\}_{E^*} \circ P^*}$$

for all $F, G \in C^\infty(D^*)$

The induced bivector Λ_{nh,D^*} is

$$\boxed{\Lambda_{nh,D^*} = \rho_a^\mu \frac{\partial}{\partial y_a} \wedge \frac{\partial}{\partial x^\mu} + \frac{1}{2} C_{ab}^c y_c \frac{\partial}{\partial y_a} \wedge \frac{\partial}{\partial y_b}}$$

That is,

$$\boxed{\{x^\mu, x^\eta\}_{nh,D^*} = 0, \qquad \{y_a, x^\mu\}_{nh,D^*} = \rho_a^\mu, \qquad \{y_a, y_b\}_{nh,D^*} = C_{ab}^c y_c}$$

Λ_{nh,D^*} is a linear bivector on D^*, but in general, does not satisfy Jacobi identity. So, we are in the case considered in the very beginning.

Particular cases

1. $E = TM$. Then the linear Poisson structure on $E^* = T^*M$ is the canonical symplectic structure. Thus, D is a distribution on M and $\{\ ,\ \}_{nh,D^*}$ is the nonholonomic bracket studied by A.J. Van der Schaft, B.M. Maschke, and others.

2. $E = \mathfrak{g}$, where $\mathfrak{g}$ is a Lie algebra. E is a Lie algebroid over a single point (the anchor map is the zero map). In this case, the linear Poisson structure on $E^* = \mathfrak{g}^*$ is the $\pm$ Lie-Poisson structure. Thus, if $D = \mathfrak{h}$ is a vector subspace of $\mathfrak{g}$, we obtain that the nonholonomic bracket (nonholonomic Lie-Poisson bracket) is given by

$$\{F, G\}_{nh,D^*\pm}(\mu) = \pm \left\langle \mu, P\left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right] \right\rangle$$

for $\mu \in \mathfrak{h}^*$, and $F, G \in C^\infty(\mathfrak{h}^*)$. In adapted coordinates

$$\{y_a, y_b\}_{nh,D^*\pm} = \pm C_{ab}^c y_c$$

3. $E =$ the Atiyah algebroid **associated with a principal G-bundle $\pi : Q \longrightarrow Q/G$**

$$E = TQ/G$$

The linear Poisson structure on $E^* = T^*Q/G$ is characterized by the following condition: "the canonical projection $T^*Q \longrightarrow T^*Q/G$ is a Poisson epimorphism"

"the Hamilton-Poincare bracket on T^*Q/G"

(See J.P. Ortega and T. S. Ratiu : Momentum maps and Hamiltonian reduction, Progress in Math., 222 Birkhauser, Boston 2004)

D a G-invariant distribution on $Q \Rrightarrow D/G$ is a vector subbundle of $E = TQ/G$

Thus, we obtain a reduced non-holonomic bracket $\{\ ,\ \}_{nh,D^*/G}$ (the non-holonomic Hamilton-Poincaré bracket on D^*/G)

We return to the general case

Taking the hamiltonian function $H : E^* \longrightarrow \mathbb{R}$ **defined by**

$$H(x^\mu, y_\alpha) = \frac{1}{2}\sum_\alpha (y_\alpha)^2 + V(x^\mu)$$

then we induce a hamiltonian function $h : D^* \longrightarrow \mathbb{R}$ **by taking** $h = H \circ P^*$**. In coordinates,**

$$h(x^\mu, y_a) = \frac{1}{2}\sum_a (y_a)^2 + V(x^\mu)$$

The nonholonomic dynamics is determined on D^* by the linear bivector Λ_{nh,D^*} and the hamiltonian function $h : D^* \longrightarrow \mathbb{R}$, that is

$$\dot{F} = \{F, h\}_{nh,D^*}$$

or, in coordinates, by the equations

$$\begin{aligned} \dot{x}^\mu &= \rho_a^\mu \frac{\partial h}{\partial y_a} = \rho_a^\mu y_a \\ \dot{y}_a &= -C_{ab}^c y_c \frac{\partial h}{\partial y_b} - \rho_a^\mu \frac{\partial h}{\partial x^\mu} \\ &= -C_{ab}^c y_c y_b - \rho_a^\mu \frac{\partial V}{\partial x^\mu} \end{aligned}$$

$$\Downarrow$$

we can apply Hamilton-Jacobi theory to nonholonomic mechanics!

Extensions to Classical Field Theories

- **M. de León, J.C. Marrero, D.M. de Diego: A geometric Hamilton-Jacobi theory for classical feld theories. in:** *Variations, Geometry and Physics in honour of Demeter Krupkas sixtyfifth birthday*, **O. Krupkova and D. J. Saunders (Editors), Nova Sci. Publ., New York, 2009, pp. 129-140.**
- **M. de León: Hamilton-Jacobi theory on the space of solutions. In preparation.**

- **L. Vitagliano, The Hamilton-Jacobi Formalism for Higher Order Field Theories,** *Int. J. Geom. Meth. Mod. Phys.* **7 (2010) 1413-1436; e-print: arXiv:1003.5236.**
- **L. Vitagliano, Hamilton-Jacobi Diffieties,** *J. Geom. Phys.* **(2011); e-print: arXiv:1104.0162.**

Other extensions

- **M. de León, J.C. Marrero, D.M. de Diego: A Hamilton-Jacobi theory for singular lagrangian systems. Preprint.**
- **M. de León: A unified Hamilton-Jacobi theory. Preprint.**

Applications

M. Barbero-Liñán, M. de León, J.C. Marrero, D.M. de Diego, M. Muñoz-Lecanda: Kinematic reduction and the Hamilton-Jacobi equation.

A close relationship between the classical Hamilton- Jacobi theory and the kinematic reduction of control systems by decoupling vector fields is shown in this paper. The geometric interpretation of this relationship relies on new mathematical techniques for mechanics defned on a skew-symmetric algebroid. This geometric structure allows us to describe in a simplifed way the mechanics of nonholonomic systems with both control and external forces.

A *skew-symmetric algebroid structure* on the vector bundle $\tau_D : D \to M$ is a $\mathbb{R}$-linear bracket $[\![\cdot,\cdot]\!]_D : \Gamma(\tau_D) \times \Gamma(\tau_D) \to \Gamma(\tau_D)$ on the space $\Gamma(\tau_D)$ of sectiond of τ_D and a vector bundle morphism $\rho_D : D \to TM$, so-called *anchor map*, such that:

1. $[\![\cdot,\cdot]\!]_D$ **is skew-symmetric, that is,**

$$[\![X,Y]\!]_D = -[\![Y,X]\!]_D, \quad \textbf{for } X,Y \in \Gamma(\tau_D).$$

2. If we also denote by $\rho_D : \Gamma(\tau_D) \to \mathfrak{X}(M)$ **the morphism of** $C^\infty(M)$**-modules induced by the anchor map then**

$$[\![X,fY]\!]_D = f[\![X,Y]\!]_D + \rho_D(X)(f)Y, \quad \textbf{for } X,Y \in \Gamma(\tau_D) \textbf{ and } f \in C^\infty(M).$$

If the bracket $[\![\cdot,\cdot]\!]_D$ satisfies the Jacobi identity, we have that the pair $([\![\cdot,\cdot]\!]_D, \rho_D)$ is a *Lie algebroid structure* on the vector bundle $\tau_D : D \to M$.

If $(\llbracket\cdot,\cdot\rrbracket_D, \rho_D)$ is a skew-symmetric algebroid structure on the vector bundle $\tau_D : D \to M$, then an *almost differential* d^D of sections of $\Lambda^k\tau_{D^*}$ with $\tau_{D^*} \colon D^* \to M$ is defined as follows

$$(\mathrm{d}^D\alpha)(X_0, X_1, \ldots, X_k) = \sum_{i=0}^{k}(-1)^i\rho_D(X_i)(\alpha(X_0, \ldots, \hat{X}_i, \ldots, X_k)) + \sum_{i<j}(-1)^{i+j}\alpha(\llbracket X_i, X_j\rrbracket_D, X_0, X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k)$$

for $\alpha \in \Gamma(\Lambda^k\tau_{D^*})$ and $X_0, X_1, \ldots, X_k \in \Gamma(\tau_D)$.

In general $(\mathrm{d}^D)^2 \neq 0$. Indeed, $(\llbracket\cdot,\cdot\rrbracket_D, \rho_D)$ is a Lie algebroid structure on the vector bundle $\tau_D : D \to M$ if and only if $(\mathrm{d}^D)^2 = 0$.

Suppose that (x^i) are local coordinates on M and that $\{e_A\}$ is a local basis of the space of sections $\Gamma(\tau_D)$ then

$$\llbracket e_A, e_B\rrbracket_D = \mathcal{C}_{AB}^{C} e_C, \quad \rho_D(e_A) = (\rho_D)_A^i \frac{\partial}{\partial x^i}.$$

The functions $\mathcal{C}_{AB}^{C}, (\rho_D)_A^i \in C^\infty(M)$ are called the *local structure functions* of the skew-symmetric algebroid on $\tau_D : D \to M$.

Let $e^A \in \Gamma(\tau_{D^*})$**, where** $\tau_{D^*} : D^* \longrightarrow M$**. If** $\{e^A\}$ **is the dual basis of** $\{e_A\}$ **then**

$$\begin{aligned} \mathrm{d}^D F &= (\rho_D)^i_A \frac{\partial F}{\partial x^i} e^A, \\ \mathrm{d}^D \kappa &= \left\{ (\rho_D)^i_A \frac{\partial \kappa_B}{\partial x^i} - \frac{1}{2} \mathcal{C}^C_{AB} \kappa_C \right\} e^A \wedge e^B, \end{aligned}$$

where $F \in C^\infty(M)$ **and** $\kappa = \kappa_B e^B \in \Gamma(\tau_{D^*})$**.**

A ρ_D***-admissible curve*** **is a curve** $\gamma : I \subseteq \mathbb{R} \longrightarrow D$ **such that**

$$\frac{\mathrm{d}(\tau_D \circ \gamma)}{\mathrm{d}t}(t) = \rho_D(\gamma(t)) \,.$$

Given a section $X \in \Gamma(\tau_D)$ **the curves** $\sigma : I \subseteq \mathbb{R} \to M$ **which verify the equation:**

$$\dot{\sigma} = \rho_D(X) \circ \sigma$$

are called *integral curves* of the section X**, that is, they are the integral curves of the associated vector field** $\rho_D(X) \in \mathfrak{X}(M)$**. Observe that** $X \circ \sigma$ **is a** ρ_D**-admissible curve. Locally, the integral curves are characterized as the solutions of the following system of differential equations**

$$\dot{x}^i = (\rho_D)^i_A X^A(x)$$

where $X = X^A e_A$.

Consider now the vector space over $\mathbb{R}$

$$H^0(d^D) = \{f \in C^\infty(Q) \mid \mathrm{d}^D f = 0\}.$$

If M is connected and D is a transitive skew-symmetric algebroid, that is,

$$\rho_D(D_q) = T_q Q, \quad \textbf{for all } q \in Q$$

then

$$H^0(\mathrm{d}^D) \simeq \mathbb{R}. \tag{16}$$

It is important to stress that condition (??) holds if the skew-symmetric algebroid is completely nonholonomic with connected base space.

Levi-Civita connection on a skew-symmetric algebroid with a bundle metric

Let $\mathcal{G}^D : D \times_M D \to \mathbb{R}$ be a bundle metric on a skew-symmetric algebroid $(D, [\![\cdot,\cdot]\!]_D, \rho_D)$. Given this bundle metric we can construct a unique connection $\nabla^{\mathcal{G}^D}$ on D which is torsion-less and metric with respect to $\mathcal{G}$. The following construction mimics the classical construction of the Levi-Civita connection for a riemannian metric on a differentiable manifold.

We will denote by $\flat_{\mathcal{G}^D} : D \to D^*$ the vector bundle isomorphism induced by $\mathcal{G}^D$ and by $\#_{\mathcal{G}^D} : D^* \to D$ the inverse morphism. The bundle metric can be locally written as $\mathcal{G}^D = (\mathcal{G}^D)_{AB} e^A \wedge e^B$.

The *Levita-Civita connection* $\nabla^{\mathcal{G}^D} : \Gamma(\tau_D) \times \Gamma(\tau_D) \to \Gamma(\tau_D)$ associated to the bundle metric $\mathcal{G}^D$ is defined by the formula:

$$\begin{aligned} 2\mathcal{G}^D(\nabla^{\mathcal{G}^D}_X Y, Z) \; = \; & \rho_D(X)(\mathcal{G}^D(Y,Z)) + \rho_D(Y)(\mathcal{G}^D(X,Z)) \\ & -\rho_D(Z)(\mathcal{G}^D(X,Y)) + \mathcal{G}^D(X, [\![Z,Y]\!]_D) \\ & +\mathcal{G}^D(Y, [\![Z,X]\!]_D) - \mathcal{G}^D(Z, [\![Y,X]\!]_D) \end{aligned}$$

for $X, Y, Z \in \Gamma(\tau_D)$.

Alternatively, $\nabla^{\mathcal{G}^D}$ is determined by

$$[\![X, Y]\!]_D = \nabla^{\mathcal{G}^D}_X Y - \nabla^{\mathcal{G}^D}_Y X \text{ (symmetry)}$$
$$\rho_D(X)(\mathcal{G}^D(Y,Z)) = \mathcal{G}^D(\nabla^{\mathcal{G}^D}_X Y, Z) + \mathcal{G}^D(Y, \nabla^{\mathcal{G}^D}_X Z) \text{ (metricity)} ,$$

The *Christoffel symbols* of the connection $\nabla^{\mathcal{G}^D}$ are given by

$$\nabla^{\mathcal{G}^D}_{e_B} e_C = \Gamma^A_{BC} e_A.$$

Assuming that the basis $\{e_A\}$ is $\mathcal{G}^D$-orthogonal, it is easy to deduce that

$$\Gamma^A_{BC} = \frac{1}{2}\left(\mathcal{C}^C_{BA} + \mathcal{C}^B_{AC} + \mathcal{C}^A_{BC}\right) \ . \tag{17}$$

Given the bundle metric $\mathcal{G}^D$, the associated *symmetric product* is defined as follows:

$$\langle X : Y \rangle_{\mathcal{G}^D} = \nabla^{\mathcal{G}^D}_X Y + \nabla^{\mathcal{G}^D}_Y X \ , \quad X, Y \in \Gamma(\tau_D) \ .$$

The symmetric product is a fundamental tool in controllability results, kinematic reduction of mechanical systems and in the characterization of geodesic invariance of distributions.

Given a bundle endomorphism $\mathcal{F} : D \to D$ (that is, $\tau_D \circ \mathcal{F} = \tau_D$) we define a *newtonian system* as the following triple $(D, \mathcal{G}^D, \mathcal{F})$. The newtonian system induces the system of differential equations:

$$\nabla^{\mathcal{G}^D}_{\gamma(t)} \gamma(t) = \mathcal{F}(\gamma(t)), \qquad t \in I \,, \tag{18}$$

where the solutions are curves $\gamma : I \subseteq \mathbb{R} \to D$ which are ρ_D-admissible.

$\mathcal{F}$ could be given by a section $F \in \Gamma(\tau_D)$ such that $\mathcal{F} = F \circ \tau_D$. An interesting particular case is when we have a potential function $V : M \to \mathbb{R}$. Then F is the section $-\mathbf{grad}_{\mathcal{G}^D} V \in \Gamma(\tau_D)$ and it is characterized by the following property

$$\mathcal{G}^D(\mathbf{grad}_{\mathcal{G}^D} V, X) = \rho_D(X)(V), \quad \textbf{for every } X \in \Gamma(\tau_D).$$

In particular, the solutions of the mechanical problem with Lagrangian $L : D \longrightarrow \mathbb{R}$:

$$L(v) = \frac{1}{2} \mathcal{G}^D(v, v) - V(\tau_D(v)) \tag{19}$$

(which are characterized by the Euler-Lagrange equations), are precisely the solutions of the newtonian system $(D, \mathcal{G}^D, F = -\mathbf{grad}_{\mathcal{G}^D} V)$.

Therefore, these solutions are ρ_D-admissible curves $\gamma : I \longrightarrow D$ such that

$$\nabla^{\mathcal{G}^D}_{\gamma(t)}\gamma(t) + \mathbf{grad}_{\mathcal{G}^D} V(\tau_D(\gamma(t))) = 0. \tag{20}$$

Given local coordinates (x^i, y^A) associated with the basis $\{e_A\}$ of sections of D the Equations (??) can be written as

$$\begin{array}{rcl} \dot{x}^i &=& (\rho_D)^i_A y^A \\ \dot{y}^C &=& -\Gamma^C_{AB} y^A y^B + \mathcal{F}^C(x,y). \end{array} \tag{21}$$

where $\mathcal{F}(x^j, y^B) = (x^j, \mathcal{F}^A(x^j, y^B))$.

Observe that Equations (??) are the equations of the integral curves of the vector field $\xi_{\mathcal{G}^D,F}$ on D, which local expression is

$$\xi_{\mathcal{G}^D,F} = (\rho_D)^i_A y^A \frac{\partial}{\partial x^i} + \left(-\Gamma^C_{AB} y^A y^B + \mathcal{F}^C\right) \frac{\partial}{\partial y^C} \ .$$

In the particular case of the mechanical system $(D, \mathcal{G}^D, V)$ the equations are:

$$\begin{array}{rcl} \dot{x}^i &=& (\rho_D)^i_A y^A \\ \dot{y}^C &=& -\Gamma^C_{AB} y^A y^B - (\mathcal{G}^D)^{CB} (\rho_D)^i_B \dfrac{\partial V}{\partial x^i} \ . \end{array}$$

where $(\mathcal{G}^D)^{AB}$ are the entries of the inverse matrix of $((\mathcal{G}^D)_{AB})$.

Example 1.

If $D = TM$, $[\![\, ,]\!]_D = [\, ,]$ the standard Lie bracket on M and $\rho_D = \mathrm{Id}_{TM}$ and $\mathcal{G}^D$ is a riemannian metric on M, then Equations (??) are the classical *Euler-Lagrange equations* for the mechanical lagrangian $L : TM \longrightarrow \mathbb{R}$.

Example 2

Given a regular distribution D on TM and a riemannian metric $\mathcal{G}^{TM}$ we may consider the riemannian orthogonal decomposition $TM = D \oplus D^{\perp}$ and the associated orthogonal projectors $P : TM \to D$ and $Q : TM \to D^{\perp}$. Denote also by $i_D : D \hookrightarrow TM$ the canonical inclusion. We induce by restriction a bundle metric $\mathcal{G}^D : D \times_M D \to \mathbb{R}$ and an skew-symmetric algebroid structure on D as follows:

$$[\![X, Y]\!]_D = P[i_D(X), i_D(Y)] \ , \quad \rho_D(X) = i_D(X)$$

where $X, Y \in \Gamma(\tau_D)$. In this case X, Y are vector fields on M taking values on D.

For this particular skew-symmetric algebroid structure Equations (??) correspond with the equations of the nonholonomic system determined by the constraints induced the distribution D and the mechanical lagrangian (??). These equations are also called in the literature *Lagrange-D'Alembert's equations*.

In coordinates, consider a basis of $\mathcal{G}^D$-orthogonal vector fields $\{X_A, X_\alpha\}$, $1 \leq A \leq m = \operatorname{rank} D$, $m+1 \leq \alpha \leq n = \dim M$ adapted to the decomposition $TM = D \oplus D^{\perp}$; that is, $D_x =$ span $\{X_A(x)\}$ and $D_x^{\perp} =$ span $\{X_\alpha(x)\}$. Observe that, in this adapted basis, the

nonholonomic constraints are rewritten as $y^\alpha = 0$, $m+1 \leq \alpha \leq n$. Therefore, the skew-symmetric structure induced on the vector subbundle $D \to M$ is locally described by:

$$
\begin{aligned}
[\![X_A, X_B]\!]_D &= P[X_A, X_B] = P(\mathcal{C}_{AB}^C X_C + \mathcal{C}_{AB}^\beta X_\beta) = \mathcal{C}_{AB}^C X_C \\
\rho_D(X_A) &= X_A \, .
\end{aligned}
$$

The *Lagrange-D'Alembert's equations* are

$$
\begin{aligned}
\dot{x}^i &= (\rho_D)_A^i y^A \\
\dot{y}^C &= -\Gamma_{AB}^C y^A y^B - (\mathcal{G}^D)^{CB} (\rho_D)_B^i \frac{\partial V}{\partial x^i} \, .
\end{aligned} \tag{22}
$$

where $X_A = (\rho_D)_A^i \frac{\partial}{\partial x^i}$, $1 \leq A \leq m$ and

$$
\Gamma_{BC}^A = \frac{1}{2} \left(\mathcal{C}_{BA}^C + \mathcal{C}_{AC}^B + \mathcal{C}_{BC}^A \right) \, .
$$

PROPOSITION.

Let $(D, [\![\ ,\ ]\!]_D, \rho_D)$ **be a skew-symmetric algebroid and consider a mechanical problem determined by** $(D, \mathcal{G}^D, V)$**. Take an arbitrary section** $X \in \Gamma(\tau_D)$ **then, the following conditions are equivalent:**

1. **If** $\sigma : I \longrightarrow M$ **is an integral curve of the vector field** $\rho_D(X)$ **that is,**

$$\dot{\sigma}(t) = \rho_D(X)(\sigma(t)), \tag{23}$$

then $\gamma = X \circ \sigma : I \longrightarrow D$ **is a solution of**

$$\nabla^{\mathcal{G}^D}_{\gamma(t)} \gamma(t) + \mathrm{grad}_{\mathcal{G}^D} V(\tau_D(\gamma(t))) = 0.$$

2. X **satisfies**

$$\nabla^{\mathcal{G}^D}_X X + \mathrm{grad}_{\mathcal{G}^D} V = 0.$$

Specializing the above Proposition to vector fields X verifying an extra condition $\mathrm{d}^D(\flat_{\mathcal{G}}(X)) = 0$ we obtain a new expression of this Proposition. Indeed, the following Theorem gives the classical expression of the Hamilton-Jacobi equation for the hamiltonian function $h : D^* \longrightarrow \mathbb{R}$:

$$h(\kappa, \kappa) = \mathcal{G}^{D^*}(\kappa, \kappa) + V(\tau_{D^*}(\kappa))$$

where $\kappa \in D^*$ and $\mathcal{G}^{D^*} : D^* \times_M D^* \longrightarrow \mathbb{R}$ is the induced fibred metric on the dual space.

THEOREM.-

Let $(D, [\![\ ,\ ]\!]_D, \rho_D)$ **be a skew-symmetric algebroid and consider a mechanical problem determined by** $(D, \mathcal{G}^D, V)$**. Take a section** $X \in \Gamma(\tau_D)$ **such that** $d^D(\flat_{\mathcal{G}}(X)) = 0$**. The following conditions are equivalent:**

1. If $\sigma : I \longrightarrow M$ **is an integral curve of the vector field** $\rho_D(X)$

$$\dot{\sigma}(t) = \rho_D(X)(\sigma(t)),$$

then $\gamma = X \circ \sigma : I \longrightarrow D$ **is a solution of**

$$\nabla^{\mathcal{G}^D}_{\gamma(t)}\gamma(t) + \mathbf{grad}_{\mathcal{G}^D} V(\tau_D(\gamma(t))) = 0.$$

2. X **satisfies** ***the Hamilton-Jacobi differential equation***

$$\mathrm{d}^D(\frac{1}{2}\mathcal{G}^D(X, X) + V) = 0. \qquad (24)$$

If $(D, [\![\cdot,]\!]_D, \rho_D)$ **is completely nonholonomic and** M **is connected or if** $H^0(d^D) \simeq \mathbb{R}$**, then Equation (??) is equivalent to**

$$\frac{1}{2}\mathcal{G}^D(X, X) + V = \mathbf{constant}.$$

Now, we will check when there exists a solution of $\mathrm{d}^D(\flat_{\mathcal{G}^D}(X)) = 0$ **of the type** $X = \mathbf{grad}_{\mathcal{G}^D} f$ **where** $f : M \to \mathbb{R}$**. Observe that, in this particular case,** $\mathrm{d}^D(\flat_{\mathcal{G}^D}(X)) = 0$ **is equivalent to** $(\mathrm{d}^D)^2(f) = 0$**. After some straightforward calculations, this condition is equivalent to:**

$$[\rho_D(Y), \rho_D(Z)](f) = (\rho_D[\![Y, Z]\!]_D)(f)$$

Condition that it is always true for the case when the bracket $[\![\cdot, \cdot]\!]_D$ **satisfies the Jacobi identity, that is, the pair** $([\![\cdot, \cdot]\!]_D, \rho_D)$ **is a** ***Lie algebroid structure*** **on the vector bundle** $\tau_D : D \to M$**.**

Extensions to Classical Field Theories

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