Multisymplectic Geometry and Infinite Jets

strong homotopy structures in classical field theory

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Introduction

The geometry underlying mechanics is symplectic geometry;
a symplectic form $\Longrightarrow$ Lie algebra of observables.

The geometry underlying field theory is multisymplectic geometry; $\star$
a multisymplectic form $\Longrightarrow$ SH Lie algebra of observables [r10,z10].

The aim of this work (in progress) is to reinterpret the result of [r10]

within the homological theory of PDEs on infinite jet spaces.

Strong Homotopy Algebraic Structures

Consider a complex of vector spaces (V, δ) and let $\mathscr{A}$ be an algebraic structure (associative, Lie, differential algebra, etc.)

Definition: roughly speaking, a SH $\mathscr{A}$-structure in (V, δ) is

a (differential) linear operation in (V, δ) which is of the type $\mathscr{A}$ only up to coherent homotopies.

Example: a SH Lie algebra structure in (V, δ) is

a (differential) skew-symmetric, bilinear product $\{\cdot, \cdot\}$ in (V, δ) s.t.

$$\{x, \{y, z\}\} + (-)^{\star}\{x, \{y, z\}\} + (-)^{\star}\{z, \{x, y\}\} = [\delta, \{\cdot, \cdot, \cdot\}](x, y, z)$$

$+$ more identities involving higher homotopies $\{\cdot, \ldots, \cdot\}_k : \Lambda^k V \longrightarrow V$.

Remark: A SH $\mathscr{A}$-structure in (V, δ) determines

an $\mathscr{A}$-structure in $H(V, \delta)$. The converse is often true also! $\star$

A SH Lie Algebra in Multisymplectic Geometry

Definition: An *n-plectic structure* on a manifold M is

a non-degenerate $(n+1)$-form ω on M, such that $d\omega = 0$. $\star$

Let (M, ω) be an n–plectic manifold.

Definition: An $(n-1)$-form σ on M is *Hamiltonian*, $\sigma \in \mathrm{Ham}$,

iff there exists a (unique) vector field $X(\sigma)$ such that $i_{X(\sigma)}\omega = d\sigma$. $\star$

$$\mathrm{Ham} \times \mathrm{Ham} \ni (\sigma, \tau) \longmapsto i_{X(\sigma)} i_{X(\tau)}\omega \in \mathrm{Ham}$$

is a well defined, skew-symmetric, bilinear map $\star$

Consider $(V, \delta) := 0 \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \longrightarrow \Omega^{n-2}(M) \xrightarrow{d} \mathrm{Ham} \longrightarrow 0$

Theorem (Rogers): There is a SH Lie algebra structure in (V, δ) given by

$$\{\sigma_1, \ldots, \sigma_k\}_k := \begin{cases} (-)^{\star} i_{X(\sigma_1)} \cdots i_{X(\sigma_k)}\omega & \text{if } \sigma_1, \ldots, \sigma_k \in \mathrm{Ham} \\ 0 & \text{otherwise} \end{cases} \quad \star$$

Let (M, ω) be a manifold with a closed $(n+1)$-form.

$$(i_Y \omega)|_L = 0, \quad \text{for all } Y \in \mathfrak{X}(M). \qquad (*)$$

Eq. $(*)$ prolong to a submanifold $\mathscr{E} \subset J^\infty(M, n)$. $\mathscr{E}$ is supplied with an involutive *"contact" distribution* $\mathscr{C}$: the machinary of *diffiety* theory applies!

Remark: Solutions of Eq. $(*)$ identify with

n-dimensional integral submanifolds of $(\mathscr{E}, \mathscr{C})$.

Definition: the *characteristic cohomology* $\overline{H}(\mathscr{E})$ of $\mathscr{E}$

is the chomology of the *horizontal de Rham complex*

$$0 \longrightarrow C^\infty(\mathscr{E}) \xrightarrow{\overline{d}} \overline{\Omega}^1 \xrightarrow{\overline{d}} \cdots \longrightarrow \overline{\Omega}^{n-1} \xrightarrow{\overline{d}} \overline{\Omega}^n \longrightarrow 0,$$

where $\overline{\Omega} := \Omega(\mathscr{E})/\mathscr{C}\Omega$, and $\mathscr{C}\Omega \subset \Omega(\mathscr{E})$ is the *"contact" ideal*.

Cohomological Theory of PDEs

Local (variational) geometric objects on the *space* **P** of solutions of $\mathscr{E}$ can be defined as cohomology of suitable complexes:

$$V\mathfrak{X} := \mathfrak{X}(\mathscr{E})/\mathscr{C} \quad \text{and} \quad V\Omega^\ell := \Lambda^\ell \mathscr{C}\Omega^1$$

Canonical *horizontal complexes* defining local geometric objects on P: ⋆

$$\mathbf{C}^\infty(\mathbf{P})_\bullet \; : \; 0 \longrightarrow C^\infty(\mathscr{E}) \longrightarrow \overline{\Omega}^1 \longrightarrow \cdots \longrightarrow \overline{\Omega}^i \longrightarrow \cdots$$

$$\mathfrak{X}(\mathbf{P})_\bullet \; : \; 0 \longrightarrow V\mathfrak{X} \longrightarrow \overline{\Omega}^1 \otimes V\mathfrak{X} \longrightarrow \cdots \longrightarrow \overline{\Omega}^i \otimes V\mathfrak{X} \longrightarrow \cdots$$

$$\mathbf{\Omega}^\ell(\mathbf{P})_\bullet \; : \; 0 \longrightarrow V\Omega^\ell \longrightarrow \overline{\Omega}^1 \otimes V\Omega^\ell \longrightarrow \cdots \longrightarrow \overline{\Omega}^i \otimes V\Omega^\ell \longrightarrow \cdots$$

Main properties of the canonical *horizontal cohomologies* [v09]:

- $\mathfrak{X}(\mathbf{P})_\bullet$ has a canonical Lie algebra structure $[\cdot,\cdot]$.
- There is a canonical differential $\mathbf{d}$ in $\mathbf{\Omega}(\mathbf{P})_\bullet$.
- There is a canonical $\boldsymbol{\omega} \in \mathbf{\Omega}^2(\mathbf{P})_{n-1}$, $\mathbf{d}\boldsymbol{\omega} = 0$, put $\mathfrak{g} := \ker \boldsymbol{\omega}$. ⋆
- $\mathbf{C}^\infty(\mathbf{P})_\bullet^{\mathfrak{g}}$ has a graded Lie algebra structure $\{\cdot,\cdot\}$ induced by $\boldsymbol{\omega}$.

A Rogers SH Lie Algebra on Infinite Jets?

Let $\mathfrak{g} = 0$. There is a morphism of complexes

$$\begin{tikzcd} 0 \arrow[r] & C^\infty(M) \arrow[r] \arrow[d] & \Omega^1(M) \arrow[r] \arrow[d] & \cdots \arrow[r] & \Omega^{n-2}(M) \arrow[r] \arrow[d] & \mathrm{Ham} \arrow[r] \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & C^\infty(\mathscr{E}) \arrow[r] & \overline{\Omega}^1 \arrow[r] & \cdots \arrow[r] & \overline{\Omega}^{n-2} \arrow[r] & \overline{\Omega}^{n-1} \arrow[r] & \overline{\Omega}^n \arrow[r] & 0 \end{tikzcd}$$

Question: can the Rogers SH Lie algebra be lifted to a SH Lie algebraic structure in $(\overline{\Omega}, \overline{d})$, inducing $(\mathbf{C}^\infty(\mathbf{P})_\bullet, \{\cdot,\cdot\})$ in cohomology?

A more general

Question: are there SH algebraic structures in the horizontal complexes inducing the canonical algebraic structures in cohomology?

A SH Lie Algebra in the Cohomological Theory of PDEs

Defining complex of vector fields on $\mathbf{P}$

$$0 \longrightarrow V\mathfrak{X} \xrightarrow{\overline{d}} \overline{\Omega}^1 \otimes V\mathfrak{X} \xrightarrow{\overline{d}} \cdots \longrightarrow \overline{\Omega}^{n-1} \otimes V\mathfrak{X} \xrightarrow{\overline{d}} \overline{\Omega}^n \otimes V\mathfrak{X} \longrightarrow 0$$

Theorem [LV]: a splitting

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{C} \arrow[r] & \mathfrak{X}(\mathscr{E}) \arrow[r] & V\mathfrak{X} \arrow[r] \arrow[l, bend left] & 0 \end{tikzcd}$$

determines a SH Lie algebra structure in $(\overline{\Omega} \otimes V\mathfrak{X}, \overline{d})$ with

$$\begin{array}{lll} 2) & \{X_1, X_2\} & := \quad (-)^{X_1}[[R, X_1]_{\mathrm{rn}}, X_2]_{\mathrm{rn}} + [\![X_1, X_2]\!] \\ 3) & \{X_1, X_2, X_3\} & := -(-)^{X_2}[[[R, X_1]_{\mathrm{rn}}, X_2]_{\mathrm{rn}}, X_3]_{\mathrm{rn}} \\ & & \vdots \\ k) & \{X_1, \ldots, X_k\} & := 0 \qquad\qquad \text{for } k > 3 \end{array}.$$

the Lie algebra induced in cohomology is $(\mathfrak{X}(\mathbf{P})_\bullet, [\cdot, \cdot])$.

The Derived Brackets Formalism

Remark: a splitting

$$0 \longrightarrow \mathscr{C} \longrightarrow \mathfrak{X}(\mathscr{E}) \longrightarrow V\mathfrak{X} \longrightarrow 0$$

determines an embedding

$$i : \overline{\Omega} \otimes V\mathfrak{X} \ni X \longmapsto i_X \in \{\text{derivations of } \Omega(\mathscr{E})\}$$

with values in an *abelian subalgebra*,

$$P : \{\text{derivations of } \Omega(\mathscr{E})\} \ni \nabla \longmapsto P\nabla \in \overline{\Omega} \otimes V\mathfrak{X} \star$$

Theorem [LV]: there exists a derivation Δ of $\Omega(\mathscr{E})$ such that

$$\{X_1, X_2, \ldots, X_k\} = (-)^{\star} P[\cdots[[\Delta, i_{X_1}], i_{X_2}], \cdots, i_{X_k}]$$

However it is not the Voronov [v05] case: $\Delta^2 \neq 0$, $[\ker P, \ker P] \not\subset \ker P$!

A "SH Differential Algebra" $\star$

Defining complex of differential forms on $\mathbf{P}$

$$0 \longrightarrow V\Omega \xrightarrow{\bar{d}} \overline{\Omega}^1 \otimes V\Omega \xrightarrow{\bar{d}} \cdots \longrightarrow \overline{\Omega}^{n-1} \otimes V\Omega \xrightarrow{\bar{d}} \overline{\Omega}^n \otimes V\Omega \longrightarrow 0$$

Theorem [LV]: a splitting

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{C} \arrow[r] & \mathfrak{X}(\mathscr{E}) \arrow[r] & V\mathfrak{X} \arrow[r] \arrow[l, bend left] & 0 \end{tikzcd}$$

determines a "SH differential algebra" structure in $(\overline{\Omega} \otimes V\Omega, \bar{d})$ with

$$\begin{array}{lll} 2) & d_2 & := 2i_R + d - \bar{d} \\ 3) & d_3 & := -i_R \\ & \vdots & \\ k) & d_k & := 0 \qquad \text{for } k > 3 \end{array}$$

the differential algebra induced in cohomology is $(\mathbf{\Omega}(\mathbf{P})_\bullet, \mathbf{d})$.

Conjectures

Conjecture: The above SH structures do not depend on the splitting up to isomorphisms of SH structures. *Only proved "up to the order 3"*!

The above theorems are independent of the multisymplectic geometry and actually hold for every diffiety!

Conjecture: There is a SH Lie algebra structure in $(\overline{\Omega}, \overline{d})$ such that

- it extends the Rogers SH Lie algebra
- it induces the Lie algebra $(\mathbf{C}^\infty(\mathbf{P}), \{\cdot,\cdot\})$ in cohomology;
- it is uniquely determined by a splitting as above, up to isos.

Conjecture: There are SH structures in all horizontal complexes such that

- they induce the corresponding canonical structures in cohomology;
- they are uniquely determined by a splitting as above, up to isos.

A lot of work to do!

A Few References

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[v05] T. Voronov, Higher derived brackets and homotopy algebras, *J. Pure Appl. Algebra* **202** (2005) 133–153.