Lagrangian and Hamiltonian Mechanics

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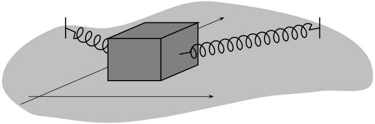

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- Γ - admissible processes: one-dimensional oriented submanifolds with border
- $\mathcal{W} : \Gamma \to \mathbb{R}$ - cost function

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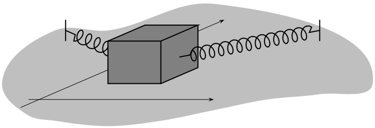

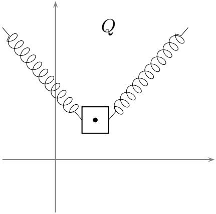

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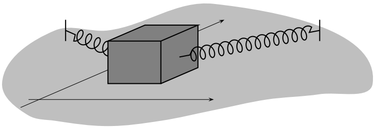

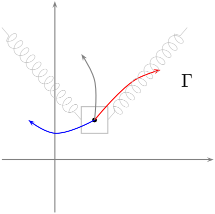

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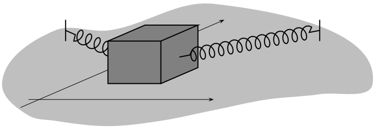

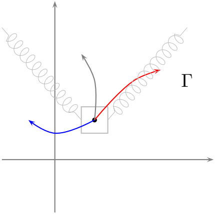

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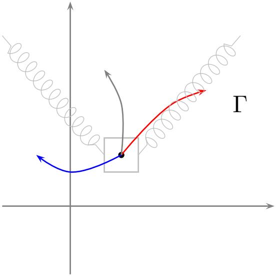

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- (Q, Δ, W) contains all the information about the system!

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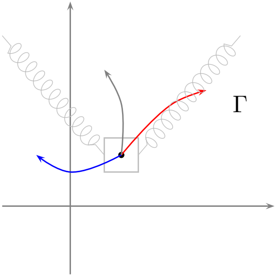

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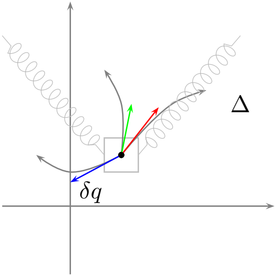

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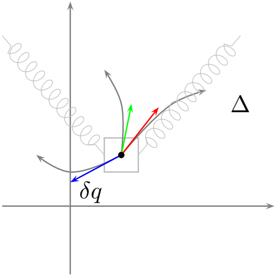

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Variational calculus in statics

In statics we are interested in finding equilibrium points of the system, isolated as well as interacting with other systems.

Definition

Point $q \in Q$ is an equilibrium point of the system if for all processes starting in q the cost function is positive, at least initially.

First order necessary condition:

$$\text{if } q \text{ is an equilibrium point than } \forall \delta q \in \Delta_q \quad W(\delta q) \geqslant 0.$$

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Interactions between systems are described by composite systems

- system (1) and (2) have the same configurations Q
- $\Delta = \Delta_1 \cap \Delta_2$
- $W = W_1 + W_2$

There are distinguished systems called regular

- not constrained: $\Delta = \mathsf{T}Q$
- the cost function is the differential of an internal energy function

$$W(\delta q) = \langle dU(q), \delta q \rangle.$$

- If q is an equilibrium point for regular system then $dU(q) = 0$

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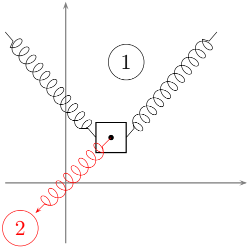

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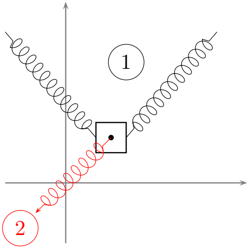

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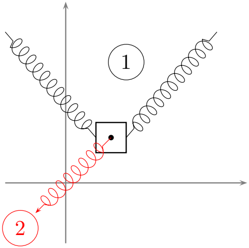

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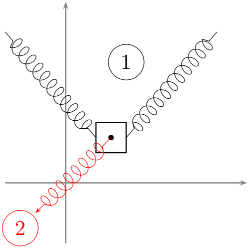

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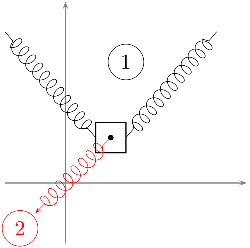

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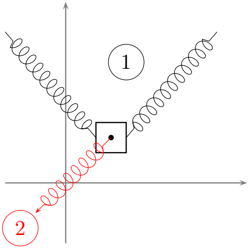

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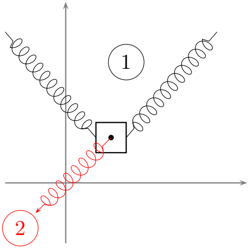

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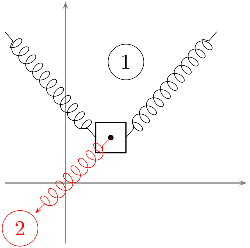

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For every point $q \in Q$ we make a list of all regular systems in equilibrium with our system.

- A regular system at q is represented by a covector $\varphi \in \mathsf{T}^*_q Q$
- If our system is in equilibrium with a regular system φ than

$$\forall \delta q \in \Delta_q \quad W(\delta q) \geqslant \langle \varphi, \delta q \rangle$$

- The subset $\mathcal{C} \subset \mathsf{T}^* Q$ of all regular systems in equilibrium with our system is called a constitutive set.
- The passage from (Q, Δ, W) to $\mathcal{C}$ is called a Legendre-Fenchel transformation.
- For large class of systems $\mathcal{C}$ contains all information of the system.
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Werner Fenchel (1905-1988)

For other theories, e.g. statics of an elastic rod, mechanics, different field theories... we need

- Configurations Q,
- Processes (or at least tangent vectors $\mathsf{T}Q$),
- Functions on Q (to define regular systems),
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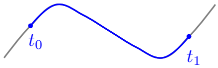
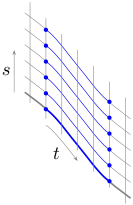
Mechanics for finite time interval

Let M be a manifold of positions of mechanical system. We will use smooth curves in M and first order Lagrangians

- Configurations: $Q = \{q : [t_0, t_1] \to M\}$.
- Functions: $S(q) = \displaystyle\int_{t_0}^{t_1} L(\dot{q})dt$.
- Curves in Q come from homotopies: $\chi : \mathbb{R}^2 \to M$
- Tangent vectors are equivalence classes of curves





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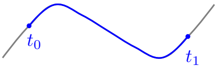

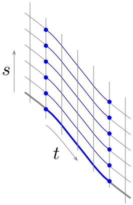

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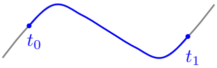

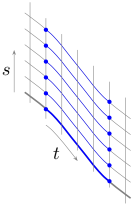

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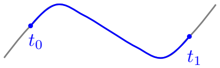

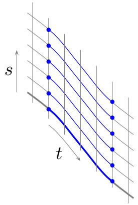

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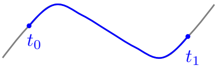

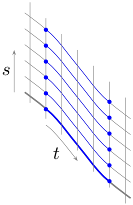

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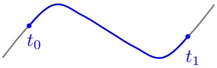

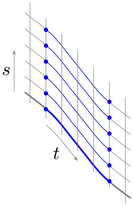

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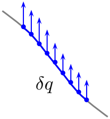

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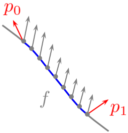

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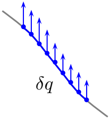

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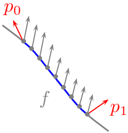

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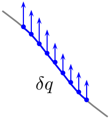

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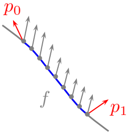

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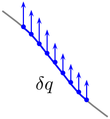

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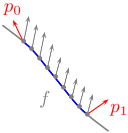

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We have found another representation of covectors (Liouville structure):

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The dynamics is a subset $\mathcal{D}$ of $\{(f, p_0, p_1)\}$

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Let M be a manifold of positions of mechanical system. We will use smooth curves in M and first order Lagrangians

- Configurations: $Q = \mathrm{T}M$, $q = \dot{x}(t)$
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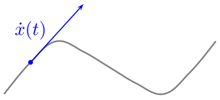

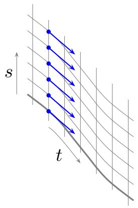

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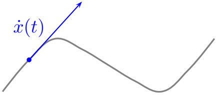

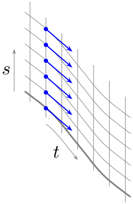

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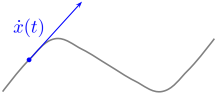

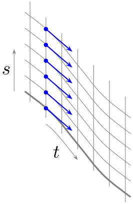

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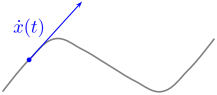

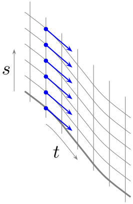

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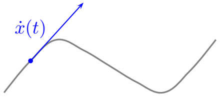

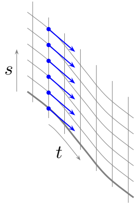

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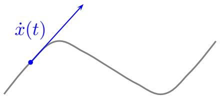

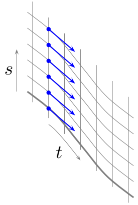

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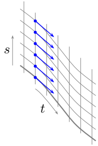

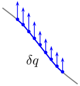

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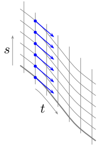

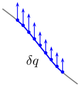

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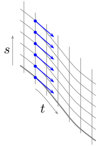

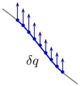

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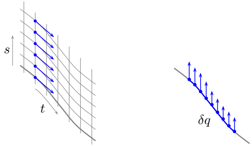

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$$[(f,p)] = \dot{p}(t) + f(t)^v$$

where $f(t)^v$ is a vector tangent to the curve $s \mapsto p(t) + sf(t)$.

- We get also the tangent evaluation between $\mathsf{T}^*\mathsf{T}M$ and $\mathsf{T}\mathsf{T}M$ defined on elemens $\dot{p}$ and $(\delta x)^{\cdot}$ with the same tangent projection δx on $\mathsf{T}M$:

$$\langle\!\langle \dot{p}, (\delta x)^{\cdot} \rangle\!\rangle = \frac{d}{dt}\langle p(t), \delta x(t)\rangle.$$

- The identification of covectors $(\mathsf{T}^*\mathsf{T}M)$ with elements $\mathsf{T}\mathsf{T}^*M$ is the map

$$\alpha_M : \mathsf{T}\mathsf{T}^*M \longrightarrow \mathsf{T}^*\mathsf{T}M$$

dual to κ_M

Lagrangian side of the Tulczyjew Triple

M - positions,
$\mathrm{T}M$ - configurations,
$L : \mathrm{T}M \to \mathbb{R}$ - Lagrangian
T^*M - phase space

$$\mathcal{D} = \alpha_M^{-1}(\mathrm{d}L(\mathrm{T}M))$$

$$\lambda : \mathrm{T}M \to \mathrm{T}^*M, \quad \lambda(v) = \xi(\mathrm{d}L(v)), \quad \lambda(x^i, \dot{x}^j) = (x^i, \frac{\partial L}{\partial \dot{x}^j}).$$

$$\mathcal{D} = \left\{ (x^i, p_j, \dot{x}^k, \dot{p}_l) : \quad p_i = \frac{\partial L}{\partial \dot{x}^i}, \quad \dot{p}_j = \frac{\partial L}{\partial x^j} \right\}$$

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Hamiltonian side of the Tulczyjew triple

There is a canonical isomorphism

$$\mathcal{R} : \mathsf{T}^*\mathsf{T}M \longrightarrow \mathsf{T}^*\mathsf{T}^*M.$$

It is an isomorphism of double vector bundles and (anti)symplectomorphism. As a symplectic relation it is generated by the canonical evaluation:

$$\mathsf{T}^*M \times_M \mathsf{T}M \ni (p, \dot{x}) \longmapsto \langle p, \dot{x} \rangle \in \mathbb{R}$$

Composed with α_M it gives another Liouville structure for $\mathsf{T}\mathsf{T}^*M$:

$$\beta_M : \mathsf{T}\mathsf{T}^*M \longrightarrow \mathsf{T}^*\mathsf{T}^*M.$$

We can therefore find another generating object for $\mathcal{D}$. Since composing symplectic relations means adding generating functions, we have

$$E(p, \dot{x}) = \langle p, \dot{x} \rangle - L(\dot{x})$$

that in some cases can be reduced to Hamiltonian function H on T^*M.

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T^*M - momenta

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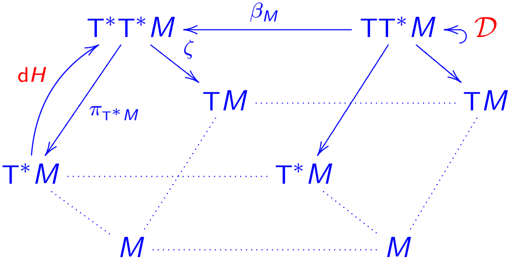

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The Tulczyjew triple for mechanics

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 & & TM \arrow[ddl, "\tau_M"] & & & TM \arrow[lll, "\tau_{T^*M}"'] \arrow[rrr] \arrow[ddl, "\tau_M"] & & & TM \arrow[ddl, "\tau_M"] \\
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 & M & & & M \arrow[lll] \arrow[rrr] & & & M &
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Other triples - elastic rod

Let M be a manifold. We use smooth curves in M and second order Lagrangians:

$$L : \mathrm{T}^2 M \ni \ddot{x} \longmapsto L(\ddot{x}) \in \mathbb{R}.$$

We concentrate on infinitesimal picture.

- Configurations $Q = \mathrm{T}^2 M$, $q = \ddot{x}(t)$. The set of configurations is a manifold, therefore

$$\mathrm{T}Q = \mathrm{T}\mathrm{T}^2 M \qquad \mathrm{T}^*Q = \mathrm{T}^*\mathrm{T}^2 M.$$

- We observe that there is the natural identification of $\mathrm{T}^2 M$ as a subset of $\mathrm{T}\mathrm{T}M$:

$$\mathrm{T}^2 M \ni (x^i, \dot{x}^j, \ddot{x}^k) \longmapsto (x^i, \dot{x}^j, \dot{x}^k, \ddot{x}^l) \in \mathrm{T}\mathrm{T}M$$

We can therefore treat the second order theory as a first order theory on $\mathrm{T}M$ with constraint

$$\{(x^i, \dot{x}^j, x'^k, \dot{x}'^l) \in \mathrm{T}\mathrm{T}M : \quad \dot{x}^j = x'^k\}$$

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We look for the convenient representation of covectors. We use calculations on finite interval:

$$\frac{d}{ds}S\circ\gamma(0)=\int_a^b\left(\frac{dL}{dx^i}-\frac{d}{dt}\frac{dL}{d\dot{x}^i}+\frac{d}{dt^2}\frac{dL}{d\ddot{x}^i}\right)]\delta x^i dt+$$
$$\left[\left(\frac{dL}{d\dot{x}^j}-\frac{d}{dt}\frac{dL}{d\ddot{x}^j}\right)\delta x^j+\frac{dL}{d\ddot{x}^k}\delta\dot{x}^k\right]_a^b$$

- "Momenta" are elements of $\mathsf{T}^*\mathsf{T}M$.
- The constitutive set: $\mathcal{D}_L$ of $\mathsf{T}^*\mathsf{T}\mathsf{T}M$ is generated by a function on submanifold.
- The convenient representation of $\mathcal{D}_L$ is

$$\mathcal{D}\subset\mathsf{T}\mathsf{T}^*\mathsf{T}M,\quad \mathcal{D}=\alpha_{\mathsf{T}M}^{-1}(\mathcal{D}_L)$$
$$\mathcal{D}=\{(x,\dot{x},\pi,\rho,x',\dot{x}',\pi',\rho'):$$
$$\pi+\rho'=\frac{dL}{d\dot{x}},\ \rho=\frac{dL}{d\ddot{x}},\ \pi'=\frac{dL}{dx},\ \dot{x}=x'\}$$

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$$\left[\left(\frac{dL}{d\dot{x}^j} - \frac{d}{dt}\frac{dL}{d\ddot{x}^j}\right)\delta x^j + \frac{dL}{d\ddot{x}^k}\delta\dot{x}^k\right]_a^b$$

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Other triples - elastic rod

The Tulczyjew triple for elastic rod:

$$
\begin{tikzcd}
& \mathsf{T}^*\mathsf{T}^*\mathsf{T}M \arrow[ddl] \arrow[dr] & & & \mathsf{T}\mathsf{T}^*\mathsf{T}M \arrow[lll, "\beta_{\mathsf{T}M}"'] \arrow[rrr, "\alpha_{\mathsf{T}M}"] \arrow[ddl] \arrow[dr] & & & \mathsf{T}^*\mathsf{T}\mathsf{T}M \arrow[ddl] \arrow[dr] & \\
& & \mathsf{T}\mathsf{T}M \arrow[ddl] & & & \mathsf{T}\mathsf{T}M \arrow[lll] \arrow[rrr] \arrow[ddl] & & & \mathsf{T}\mathsf{T}M \arrow[ddl] \\
\mathsf{T}^*\mathsf{T}M \arrow[dr] & & & \mathsf{T}^*\mathsf{T}M \arrow[lll] \arrow[rrr] \arrow[dr] & & & \mathsf{T}^*\mathsf{T}M \arrow[dr] & & \\
& \mathsf{T}M & & & \mathsf{T}M \arrow[lll] \arrow[rrr] & & & \mathsf{T}M &
\end{tikzcd}
$$

Generating object for $\mathcal{D}_H$:

$$E : \mathsf{T}^*\mathsf{T}M \times_{\mathsf{T}M} \mathsf{T}^2M \to \mathbb{R}$$

Generating object for $\mathcal{D}_L$

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\begin{tikzcd}
\color{red}{\mathcal{D}_H} \arrow[r, hook] & \mathsf{T}^*\mathsf{T}^*\mathsf{T}M \arrow[dr] \arrow[ddl] & & & \mathsf{T}\mathsf{T}^*\mathsf{T}M \arrow[lll, "\beta_{\mathsf{T}M}"'] \arrow[rrr, "\alpha_{\mathsf{T}M}"] \arrow[dr] \arrow[ddl] & & & \mathsf{T}^*\mathsf{T}\mathsf{T}M \arrow[dr] \arrow[ddl] & \color{red}{\mathcal{D}_L} \arrow[l, hook'] \\
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\mathsf{T}^*\mathsf{T}M \arrow[dr, "\pi_M"] & & & \mathsf{T}^*\mathsf{T}M \arrow[lll] \arrow[rrr] \arrow[dr] & & & \mathsf{T}^*\mathsf{T}M \arrow[dr] & & \\
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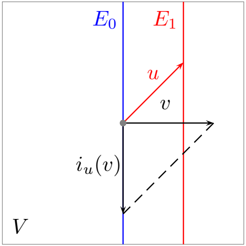
Other triples - Newtonian mechanics

The Newtonian space-time is a (N, V, τ, g), where N is a four-dimensional affine space modeled on V, $\tau \in V^*$, $g : E_0 \to E_0^*$, $E_0 = \ker \tau$.



- Homogeneous Lagrangian depends on an inertial frame u

$$L_u(x, v) = \frac{m}{2}\langle g(i_u(v)), i_u(v)\rangle - \langle \tau, v\rangle U(x)$$

- The transformation rule is an affine one:

$$L_{u'}(x, v) = L_u(x, v) + \langle \sigma(u, u'), v\rangle$$

- The frame independent Lagrangian is not a function. It is a section of the bundle $N \times W \longrightarrow N \times V$
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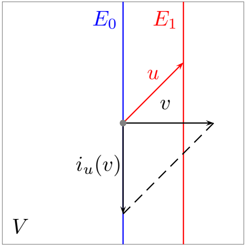

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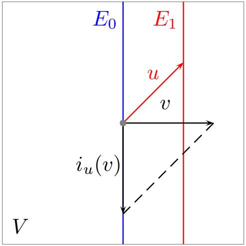

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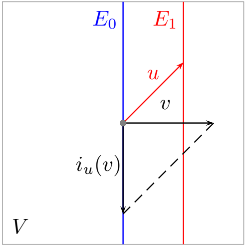

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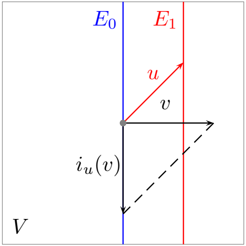

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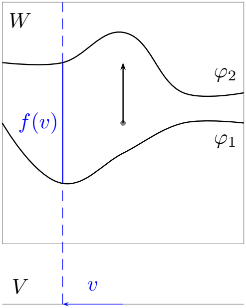

- The difference of two sections is a function $\varphi_2 - \varphi_1 = f$
- (v_1, φ_1) and (v_2, φ_1) are equivalent if $v_1 = v_2, \quad d(\varphi_2 - \varphi_1)(v) = 0$
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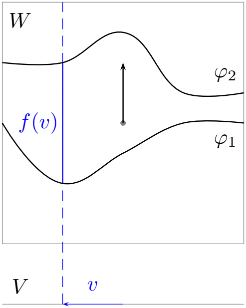

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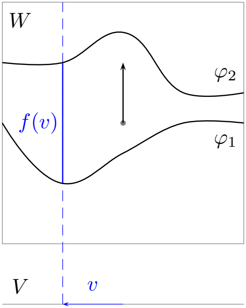

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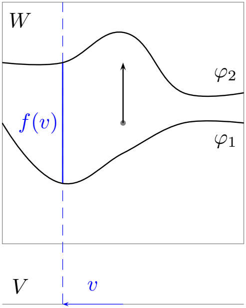

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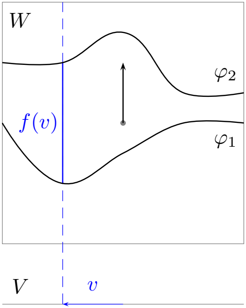

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& & N \times V \arrow[rrr, dotted, no head] \arrow[ddl, dotted, no head] & & & N \times V \arrow[ul, bend right, "\mathrm{d}L"'] \arrow[dll, "\lambda"'] \arrow[ddl, dotted, no head] \\
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There is an isomorphism

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It is possible therefore to find another Liouville structure for $\mathsf{T}(N \times \mathcal{P})$:

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 & & N \times V \arrow[rrr, dotted, no head] \arrow[ddl, dotted, no head] & & & N \times V \arrow[ddl, dotted, no head] \\
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 & \mathrm{T}^*(N \times \mathcal{P}) \arrow[dr] \arrow[ddl] & & & \mathrm{T}(N \times \mathcal{P}) \arrow[lll, "\beta"'] \arrow[dr] \arrow[ddl] & \mathcal{D} \arrow[l, hook'] \\
 & & N \times V \arrow[rrr, dotted, no head] \arrow[ddl, dotted, no head] & & & N \times V \arrow[ddl, dotted, no head] \\
N \times \mathcal{P} \arrow[uur, bend left, "\mathrm{d}H"] \arrow[rrr, dotted, no head] \arrow[dr, dotted, no head] & & & N \times \mathcal{P} \arrow[dr, dotted, no head] & & \\
 & N \arrow[rrr, dotted, no head] & & & N &
\end{tikzcd}$$

The generating object in this case is not a single Hamiltonian function but a family of Hamiltonians.

Other triples - Newtonian mechanics

There is an isomorphism

$$\mathsf{T}^*(N \times \mathcal{P}) \simeq \mathsf{P}(N \times W)$$

It is possible therefore to find another Liouville structure for $\mathsf{T}(N \times \mathcal{P})$:

$$\beta : \mathsf{T}(N \times \mathcal{P}) \longrightarrow \mathsf{T}^*(N \times \mathcal{P})$$

We get this way the Hamiltonian side of the triple:

$$\begin{tikzcd}
 & \mathsf{T}^*(N\times\mathcal{P}) \arrow[dr] \arrow[ddl] & & & \mathsf{T}(N\times\mathcal{P}) \arrow[lll, "\beta"'] \arrow[dr] \arrow[ddl] & \mathcal{D} \arrow[l, hook] \\
 & & N\times V \arrow[rrr, dotted, no head] \arrow[ddl, dotted, no head] & & & N\times V \arrow[ddl, dotted, no head] \\
N\times\mathcal{P} \arrow[uur, bend left, "\mathrm{d}H"] \arrow[rrr, dotted, no head] \arrow[dr, dotted, no head] & & & N\times\mathcal{P} \arrow[dr, dotted, no head] & & \\
 & N \arrow[rrr, dotted, no head] & & & N &
\end{tikzcd}$$

The generating object in this case is not a single Hamiltonian function but a family of Hamiltonians.

Other triples - Newtonian mechanics

The Tulczyjew triple for frame independent, homogeneous Newtonian mechanics:

$$
\begin{tikzcd}
& \mathrm{T}^*(N\times\mathcal{P}) \arrow[ddl] \arrow[dr] & & & \mathrm{T}(N\times\mathcal{P}) \arrow[lll, "\beta"'] \arrow[rrr, "\alpha"] \arrow[ddl] \arrow[dr] & & & \mathrm{P}(N\times W) \arrow[ddl] \arrow[dr] & \\
& & N\times V \arrow[ddl] & & & N\times V \arrow[lll] \arrow[rrr] \arrow[ddl] & & & N\times V \arrow[ddl] \\
N\times\mathcal{P} \arrow[dr] & & & N\times\mathcal{P} \arrow[lll] \arrow[rrr] \arrow[dr] & & & N\times\mathcal{P} \arrow[dr] & & \\
& N & & & N \arrow[lll] \arrow[rrr] & & & N &
\end{tikzcd}
$$

Conclusions

- Tulczyjew triples come from variational calculus, that is adapted for statics, but can be also used in other theories
- The triples are constructed, not postulated!

THANK YOU!

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