Extension of Sullivan models to transitive Lie algebroids

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Introduction

The de Rham theorem

It is well known that the integration map gives us an isomorphism between de Rham cohomology and the singular cohomology of a smooth manifold. Precisely, if M is a smooth manifold, the map

$$\int : \Omega^p(M) \longrightarrow C^p(M;\mathbf{R})$$

$$\int(\omega)(\alpha) = \int_{\Delta_p} \alpha^*\omega$$

induces the isomorphism in cohomology.

Introduction

The de Rham gave us a new method of study of the topology of manifolds by considering the de Rham algebra of the smooth forms. The de Rham theorem quickly originated a deep development of the topology of manifolds. As a first step of this development, D. Sullivan worked on the extension of the de Rham theorem to more general spaces, originating a preliminary Sullivan-Rham theorem.

Introduction

Piecewise forms

Let M be a smooth manifold smoothly triangulated by a simplicial complex K. Associated to this triangulation, there are two differential algebras of differential forms. One algebra is the algebra $\Omega^p_{ps}(M)$ of all piecewise smooth forms, in which a piecewise smooth form is a family of smooth forms, one on each simplex of K, which are compatible under restrictions to faces. Other algebra is the algebra $\Omega^p_{PL}(M)$ of all piece-linear smooth forms with coefficients in $\mathbb{Q}$ which is constructed in the same way as $\Omega^p_{ps}(M)$ but using now polynomial forms with coefficients in $\mathbb{Q}$ defined on each simplex of K.

Introduction

The Sullivan-Rham theorem - piecewise smooth case

Let M be a smooth manifold smoothly triangulated by a simplicial complex K. Then, the integration map

$$\int : \Omega^p_{ps}(M) \longrightarrow C^p(K; \mathbf{R})$$

$$\int(\omega)(\alpha) = \int_{\Delta_p} t^*\omega$$

where $t : |K| \longrightarrow M$ is a smooth triangulation, is the isomorphism in cohomology.

Introduction

The Sullivan-Rham theorem - piecewise linear case

Given a smooth manifold M smoothly triangulated by a simplicial complex K. Then, the integration map induces the isomorphism

$$H^*_{PL}(M) \longrightarrow H^*(M; \mathbf{R})$$

Consequently, the inclusion map

$$\Omega^*_{PL}(M) \otimes_{\mathbf{Q}} \mathbf{R} \longrightarrow \Omega^*_{ps}(M)$$

and the restriction map

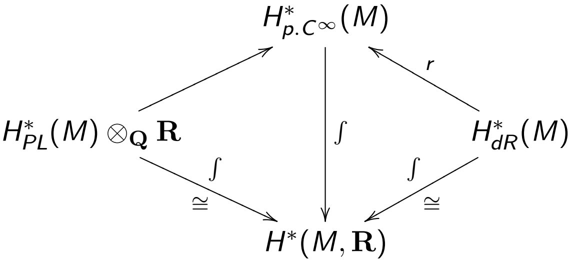
$$r : \Omega^*(M) \longrightarrow \Omega^*_{ps}(M)$$

both induce isomorphism on cohomology.

Introduction

Diagram

These theorems can be summarized in the following commutative diagram



Introduction

Models

Sullivan and other mathematicians have implemented several strategies in the study of the de Rham algebra of the smooth forms. Among them, it is the theory of the models. This theory consists in finding other graded algebras, inside the de Rham algebra of the smooth forms, such that the canonical inclusion induces an isomorphism in cohomology. Roughly, if M is a smooth manifold, a Sullivan model for M is a quasi-isomorphism

$$\psi : (S, d) \longrightarrow \Omega^*_{PL}(M)$$

where (S, d) is a free commutative differential graded algebra and the differential d satisfies the nilpotence condition and is decomposable. Since the Sullivan models are unique up to isomorphism and in order to obtain something easier, one may replace the algebra $\Omega^*_{PL}(M)$ by the de Rham algebra $\Omega^*(M)$.

Introduction

Mishchenko's Inspiration

In spite of the cohomology theory of vector bundles is much more complicated, the cohomology theory of Lie algebroids developed by Kubarski, Mackenzie and other mathematicians have reduced several obstructions and have led to very substantial progress in the study of the cohomology theory of Lie algebroids. These refinements have inspired Mishchenko and led him to conjecture, when a transitive Lie algebroid on a combinatorial manifold is given, the morphism given by restriction, which takes smooth forms with values in the Lie algebroid into piecewise smooth forms with values in the same Lie algebroid, still remains an isomorphism in cohomology.

Piecewise smooth cohomology

Definition - Sheaf of Lie algebroids

Definition - Sheaf of transitive Lie algebroids. Let K be a simplicial complex. A sheaf of transitive Lie algebroids on K is a family $(\mathcal{A}_\alpha)_{\alpha \in K}$ such that, for each $\alpha \in K$, $\mathcal{A}_\alpha$ is a transitive Lie algebroid on α and, for each simplex $\beta \in K$ such that β is a face of α, $\mathcal{A}_\beta = (\mathcal{A}_\alpha)^!_\beta$

Piecewise smooth cohomology

Definition - Piecewise smooth forms

Definition - piecewise smooth form on a sheaf. Let K be an simplicial complex and $(\mathcal{A}_\alpha)_{\alpha\in K}$ a sheaf of transitive Lie algebroids on K. A piecewise smooth form of degree p ($p \geq 0$) on the sheaf $(\mathcal{A}_\alpha)_{\alpha\in K}$ is a family $(\omega_\alpha)_{\alpha\in K}$ such that the following conditions are satisfied.

- for each $\alpha \in K$, $\omega_\alpha \in \Omega^p(\mathcal{A};\alpha)$.
- for each α, $\beta \in K$, with β face of α and $\varphi : \beta \hookrightarrow \alpha$ the inclusion, $(\varphi^!)^*\omega_\alpha = \omega_\beta$

Thus, a piecewise smooth form is a collection of smooth forms, one on each simplex of K, which are compatible under restriction to faces. The set of all piecewise smooth forms of degree p on the sheaf $(\mathcal{A}_\alpha)_{\alpha\in K}$ will be denoted by $\Omega^p_{\mathrm{ps}}(\{\mathcal{A}_\alpha\}_{\alpha\in K})$.

Piecewise smooth cohomology

Differential graded algebra of piecewise smooth forms

The operations of wedge product and a differential are defined on $\Omega^{p}_{\mathrm{ps}}(\{\mathcal{A}_\alpha\}_{\alpha\in K})$ by the corresponding operations on each α, giving to $\Omega^{p}_{\mathrm{ps}}(\{\mathcal{A}_\alpha\}_{\alpha\in K})$ a structure of differential graded algebra defined over $\mathbf{R}$. The cohomology space of this complex will be denoted by $H^{*}_{ps}(\{\mathcal{A}_\alpha\}_{\alpha\in K}); K)$.

Piecewise smooth cohomology

Extension of piecewise smooth forms

Extension lemma. Let K be a simplicial complex and $(\mathcal{A}_\alpha)_{\alpha\in K}$ a sheaf of transitive Lie algebroids on K. Let L be a simplicial subcomplex of K and consider the subsheaf of Lie algebroids $(\mathcal{A}_\alpha)_{\alpha\in L}$ on L. Then, any piecewise smooth form of degree p defined on L can be piecewise smoothly extended to a piecewise smooth form of degree p defined on the whole K.

Piecewise smooth cohomology

Mayer-Vietoris sequence for piecewise smooth forms

Mayer-Vietoris sequence. Let K be a simplicial complex and $(\mathcal{A}_\alpha)_{\alpha\in K}$ a sheaf of transitive Lie algebroids on K. Let K_0 and K_1 be two simplicial subcomplexes of K such that $K = K_0 \cup K_1$ and set $L = K_0 \cap K_1$. Then, it holds a exact short sequence of cochain complexes

$$H^{p-1}_{ps}(\mathcal{A}_{K_0\cap K_1}) \xrightarrow{F} H^p_{ps}(\mathcal{A}_K) \xrightarrow{F} H^p_{ps}(\mathcal{A}_{K_0}) \oplus H^p_{ps}(\mathcal{A}_{K_1})$$

$$\xrightarrow{=} H^p_{ps}(\mathcal{A}_{K_0}) \oplus H^p_{ps}(\mathcal{A}_{K_1}) \xrightarrow{g} H^p_{ps}(\mathcal{A}_{K_0\cap K_1}) \xrightarrow{f} H^{p+1}_{ps}(\mathcal{A}_K)$$

Piecewise smooth cohomology on combinatorial manifolds

Formulation of the problem

Let M be a smooth manifold smoothly triangulated by a simplicial complex K and $\mathcal{A}$ a transitive Lie algebroid on M. We can consider a sheaf of Lie algebroids obtained from the Lie algebroid $\mathcal{A}$ by restriction of $\mathcal{A}$ to the simplices of K. There is a natural morphism

$$\mathbf{r}^p : \Omega^p(\mathcal{A}; M) \longrightarrow \Omega^p_{\mathrm{ps}}(\mathcal{A}; K)$$

defined by

$$\omega \longrightarrow (\omega_\Delta)_{\Delta \in K}$$

Piecewise smooth cohomology on combinatorial manifolds

Main theorem

Main theorem. Let M be a smooth manifold smoothly triangulated by a simplicial complex K and $\mathcal{A}$ a transitive Lie algebroid on M. Then the map

$$r^p : \Omega^p(\mathcal{A}; M) \longrightarrow \Omega^p_{\mathrm{ps}}(\mathcal{A}; K)$$

$$\omega \longrightarrow (\omega_\Delta)_{\Delta \in K}$$

induces the isomorphism in cohomology.

Piecewise smooth cohomology on combinatorial manifolds

Main theorem

Proof of main theorem - Step 1. The complex K is finite and let $v_0, \ldots, v_N$ be the family of all vertices. Setting $U = \bigcup_{j=0}^{N-1} \mathbf{St}\,(v_j)$ and $V = \mathbf{St}\,(v_N)$ of M, we have that

$$M = U \cup V$$

and

$$U \cap V = \bigcup_{j=0}^{N-1} \mathbf{St}\,(v_j) \cap \mathbf{St}\,(v_N) = \bigcup_{j=0}^{N-1} \left(\mathbf{St}\,(v_j) \cap \mathbf{St}\,(v_N)\right) = \bigcup_{j=0}^{N-1} \mathbf{St}\,[v_j, v_N]$$

and the proof is made by induction on the number of vertices and by using the Mayer-Vietoris sequence.

Piecewise smooth cohomology on combinatorial manifolds

Main theorem

Proof of main theorem - Step 2. Let $s_1, \ldots, s_k$ be simplices of K and consider the open subsets $U_j = \mathbf{St}\,(s_j)$. For $l \in \{1, \ldots, k\}$ fixed, consider the open subsets $U = U_1 \cup \cdots \cup U_l$ and $V = U_{l+1} \cup \cdots \cup U_k$ of M and assume that $M = U \cup V$. Then we have a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^p(\mathcal{A}_M) & \stackrel{\lambda}{\longrightarrow} & \Omega^p(\mathcal{A}_U) \oplus \Omega^p(\mathcal{A}_V) & \stackrel{\mu}{\longrightarrow} & \Omega^p(\mathcal{A}_{U\cap V}) & \longrightarrow & 0 \\ & & \downarrow r & & \downarrow r & & \downarrow r & & \\ 0 & \longrightarrow & \Omega^p_{ps}(\mathcal{A}_M) & \stackrel{\delta}{\longrightarrow} & \Omega^p_{ps}(\mathcal{A}_U) \oplus \Omega^p_{ps}(\mathcal{A}_V) & \stackrel{\pi}{\longrightarrow} & \Omega^p_{ps}(\mathcal{A}_{U\cap V}) & \longrightarrow & 0 \end{array}$$

Piecewise smooth cohomology on combinatorial manifolds

Main theorem

Proof of main theorem - Step 3. Every transitive Lie algebroid over a contractible manifold is isomorphic to the trivial Lie algebroid (Mackenzie).

Piecewise smooth cohomology on combinatorial manifolds

Main theorem

Proof of main theorem - Step 4. Let s a simplex of K and $U = \mathrm{St}(s)$. Consider the trivial Lie algebroid $\mathcal{A} = TU \times \mathfrak{g}$ on U. Then, the restriction map

$$r : \Omega^*(\mathcal{A}_U) \longrightarrow \Omega^*_{ps}(\mathcal{A}_U)$$

induces isomorphism in cohomology. This is a consequence of the equalities

$$\mathrm{H}_{\mathrm{dR}}(\mathcal{A}_U) \simeq \mathrm{H}_{\mathrm{dR}}(U) \otimes \mathrm{H}(\mathfrak{g}) \quad \textbf{(Kubarski)}$$

$$\mathrm{H}_{ps}(\mathcal{A}_U) \simeq \mathrm{H}_{ps}(U) \otimes \mathrm{H}(\mathfrak{g})$$

$$H(r) = H(r^{TM}) \otimes \mathbf{Id} : H^*_{dR}(U; R) \otimes H^*(\mathfrak{g}) \longrightarrow H^*_{ps}(U; R) \otimes H^*(\mathfrak{g})$$

and applying the Rham-Sullivan theorem for the simplicial manifold U.

Piecewise smooth cohomology on combinatorial manifolds

Main theorem

References

- Jan Kubarski, *The Chern-Weil homomorphism of regular Lie algebroids*, Publ. Dep. Math. University of Lyon 1, 1991.
- Kirill Mackenzie, *Lie groupoids and Lie algebroids in differential*, Geometry, London Mathematical Society Lecture Note Series 124, Cambridge, 1987. MR 89g:58225
- Dennis Sullivan, *Infinitesimal computations in topology*, Publ. I.H.E.S. 47 (1977) 269-331.
- Hassler Whitney, *Geometric Integration Theory*, Princeton University Press, 1957.

Piecewise smooth cohomology on combinatorial manifolds

Main theorem

Thank you