Tulczyjew Triples and Dirac Algebroids

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Introduction

- Dirac structures
- Double vector bundles
- Dirac algebroids
- Examples and applications
- References

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  - W. M. Tulczyjew: "Les sous-variétés lagrangiennes et la dynamique lagrangienne" (French), *C. R. Acad. Sci. Paris Sér. A-B* **283** (1976), no. 8, Av, A675–A678.
- How to deal with nonholonomic constraints? $\rightarrow$ Dirac structures

We are looking for a geometrical framework of analytical mechanics which includes

- Reduction procedures $\rightarrow$ algebroids
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Dirac structures

- There is a canonical symmetric pairing on the extended tangent bundle (Pontryagin bundle) $\mathcal{T}N = \mathsf{T}N \oplus_N \mathsf{T}^*N$:

$$(X_1 + \alpha_1 \,|\, X_2 + \alpha_2) = \frac{1}{2}\left(\alpha_1(X_2) + \alpha_2(X_1)\right) .$$

- Courant-Dorfman bracket on the space of $Sec(\mathcal{T}N)$:

$$[\![X_1 + \alpha_1, X_2 + \alpha_2]\!] = [X_1, X_2] + \mathcal{L}_{X_1}\alpha_2 - \imath_{X_2} \mathrm{d}\alpha_1 .$$

Definition

An almost Dirac structure on the smooth manifold N is a subbundle D of $\mathcal{T}N$ which is maximally isotropic with respect to the symmetric pairing $(\cdot|\cdot)$. If, additionally, the space of sections of D is closed under the Courant-Dorfman bracket, we speak about a Dirac structure.

Note that here a subbundle D may be supported on a submanifold $N_0 \subset N$.

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Note that here a subbundle D may be supported on a submanifold $N_0 \subset N$.

- The *first integrability condition* for the almost Dirac structure says that

$$pr_{\mathsf{T}N}(D) \subset \mathsf{T}N_0\,,$$

so the Courant-Dorfman bracket reduces to a well-defined bracket $[\![\cdot,\cdot]\!]_D$ on sections of D.

- The *second integrability* condition says that $[\![\cdot,\cdot]\!]_D$ takes values in $Sec(D)$:

$$[\![\cdot,\cdot]\!]_D : Sec(D) \times Sec(D) \to Sec(D) \subset Sec((\mathcal{T}N)_{|N_0})\,.$$

By definition, an almost Dirac structure is a Dirac structure if and only if it satisfies both the integrability conditions.

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- For $\Pi \in Sec(\bigwedge^2 \mathsf{T}N)$, $\widetilde{\Pi} : \mathsf{T}^*N \ni \alpha \longmapsto \imath_\alpha \Pi \in \mathsf{T}N$,

$$D = graph(\widetilde{\Pi}) \subset \mathcal{T}N \quad \text{is an almost Dirac structure}.$$

Π is a Poisson $\Leftrightarrow$ D is a Dirac structure.

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ω is a closed $\Leftrightarrow$ D is a Dirac structure.

- For a distribution Δ on N,

$$D = \Delta \oplus \Delta^\perp \subset \mathcal{T}N \quad \text{is an almost Dirac structure}.$$

Δ is integrable $\Leftrightarrow$ D is a Dirac structure.

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Double vector bundles

Definition

A double vector bundle is a manifold with two compatible vector bundle structures. Compatibility means that the Euler vector fields (generators of homothethies) associated with the two structures commute.

- τ_1, τ_2, τ_1', τ_2' are v.b.
- The core

$$C = \{k \in K : \ \tau_1(k) = 0, \tau_2(k) = 0\},$$

  τ_0 is a v.b.
- $(\tau_1.\tau_1')$, $(\tau_2.\tau_2')$ are v.b. morphisms
- There is one more (affine) bundle

$$\tau_1 \times \tau_2 : K \longrightarrow K_1 \times_M K_2$$

  modeled on the pull-back of the core

$$K_1 \times_M K_2 \times_M C \longrightarrow K_1 \times_M K_2$$

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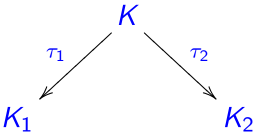

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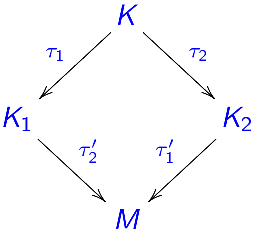

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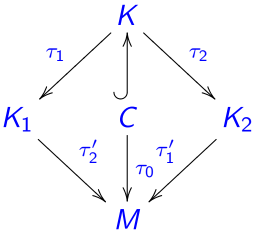

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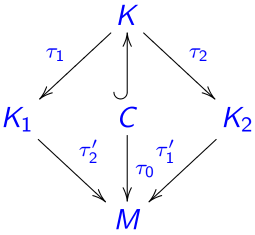

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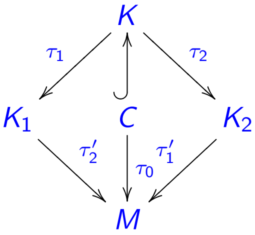

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$$\pi : E^* \longrightarrow M$$
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$$\tau_M : \mathsf{T}M \longrightarrow M$$
$$(x^a, \dot{x}^b) \longmapsto (x^a)$$

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$$\nabla_1 = \dot{x}^a \partial_{\dot{x}^a} + \dot{\xi}_i \partial_{\dot{\xi}_i}$$

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Double vector bundles

First example: $\mathsf{T}E^*$.

$$\tau : E \longrightarrow M$$
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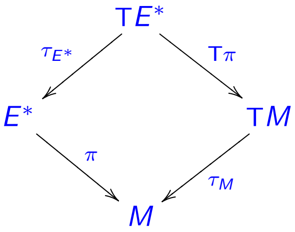

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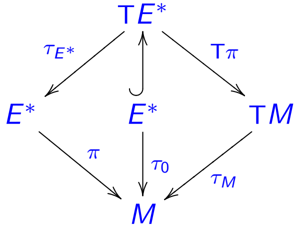

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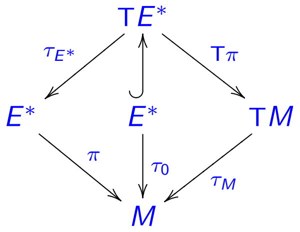

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Second example: T^*E^*.

$$\pi_{E^*} : \mathsf{T}^*E^* \longrightarrow E^*$$
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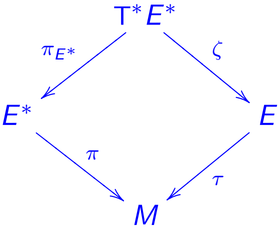

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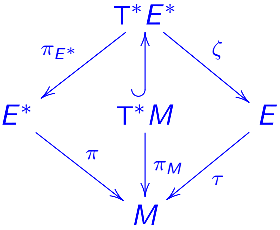

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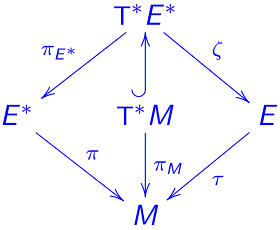

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T^*E^* is (symplectically) isomorphic to T^*E. The graph of the canonical d.v.b. anti-symplectic isomorphism $\mathcal{R}$ is the lagrangian submanifold generated in

$$\mathsf{T}^*(E^* \times E) \simeq \mathsf{T}^*E^* \times \mathsf{T}^*E \quad \text{by} \quad E^* \times_M E \ni (\xi, y) \longmapsto \xi(y) \in \mathbb{R}.$$

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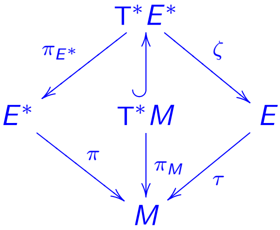

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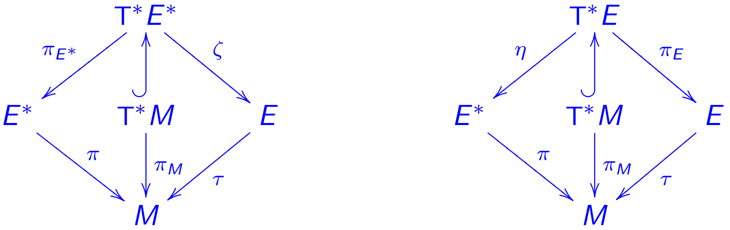

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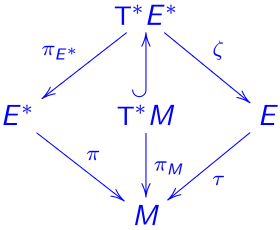

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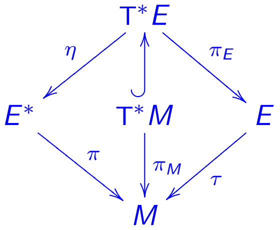

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Definition

A vector subbundle of a vector bundle $\tau : E \to M$ is a submanifold $F \subset E$ which is invariant with respect to the family of homotheties, $h_t(v) = tv$, defined by the vector bundle structure τ.

- Euler vector field is tangent to F;
- F is supported on a submanifold $M_0 \subset M$.

Definition

A double vector subbundle of a double vector bundle K is a submanifold $D \subset K$ which is invariant with respect to both families of homotheties defined by the two compatible vector bundle structures, τ_1 and τ_2,

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Dirac algebroids

Linearity of different geometrical structures is related to double vector bundle structures.

- A bivector field Π on a vector bundle F is linear if the corresponding map
$$\widetilde{\Pi} : \mathsf{T}^*F \longrightarrow \mathsf{T}F$$
is a morphism of double vector bundles.
- A two-form ω on a vector bundle F is linear if the corresponding map
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- Linear connections ...

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- A general algebroid is a double vector bundle morphism covering the identity on E^*:

$$\varepsilon(x^a, y^i, p_b, \xi_j) = (x^a, \xi_i, \rho_k^b(x)y^k, c_{ij}^k(x)y^i\xi_k + \sigma_j^a(x)p_a)$$

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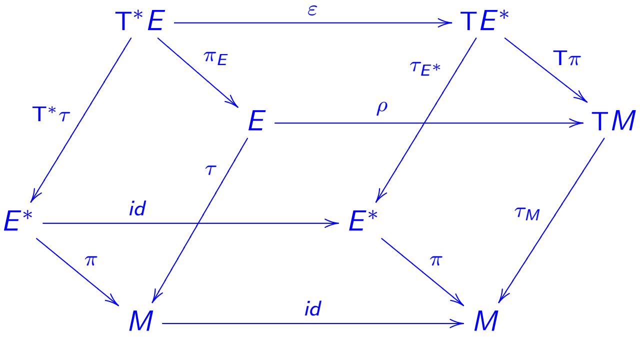

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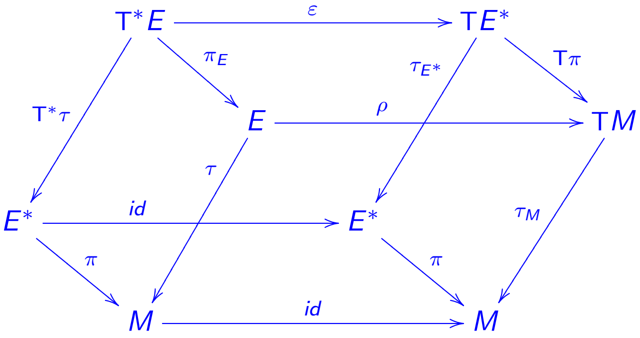

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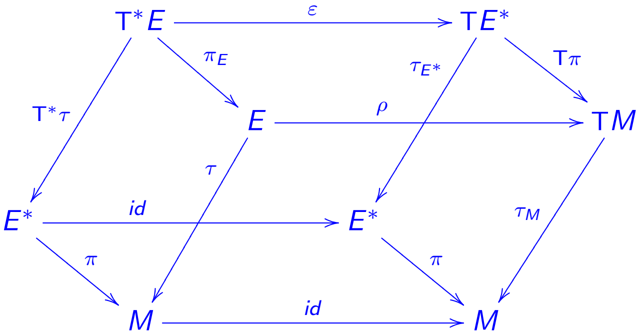

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Definition

A Dirac algebroid (resp., Dirac-Lie algebroid) structure on a vector bundle E is an almost Dirac (resp., Dirac) subbundle D of $\mathsf{T}E^*$ being a double vector subbundle, i.e., D is not only a subbundle of $\tau_1 : \mathsf{T}E^* \to E^*$ but also a vector subbundle of the vector bundle $\tau_2 : \mathsf{T}E^* \to \mathsf{T}M \oplus_M E$.

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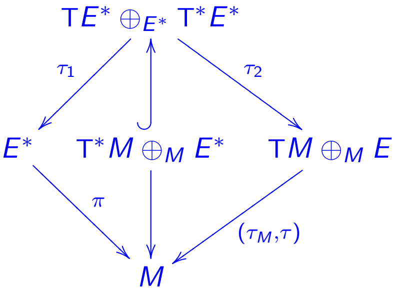

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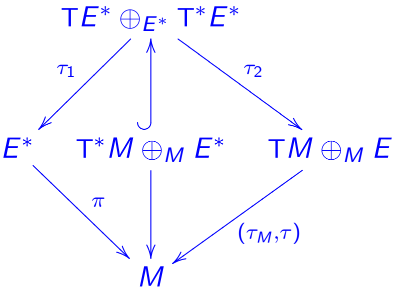

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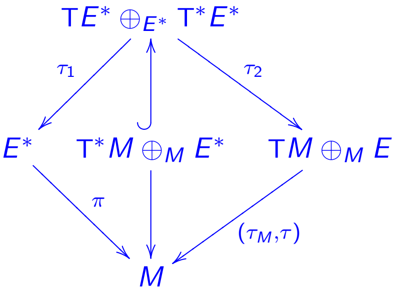

$$\tau_1 : (x^a, \xi_i, \dot{x}^b, \dot{\xi}_j, p_c, y^k) \mapsto (x^a, \xi_i),$$

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Definition

A Dirac algebroid (resp., Dirac-Lie algebroid) structure on a vector bundle E is an almost Dirac (resp., Dirac) subbundle D of $\mathcal{T}E^*$ being a double vector subbundle, i.e., D is not only a subbundle of $\tau_1 : \mathcal{T}E^* \to E^*$ but also a vector subbundle of the vector bundle $\tau_2 : \mathcal{T}E^* \to \mathsf{T}M \oplus_M E$.

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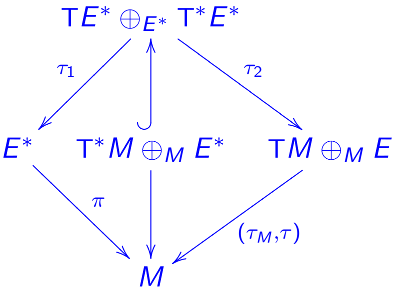

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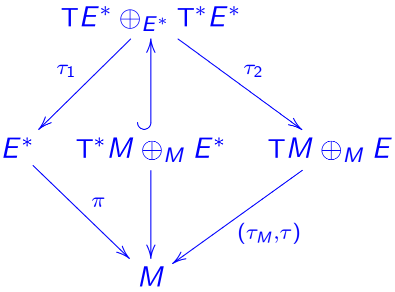

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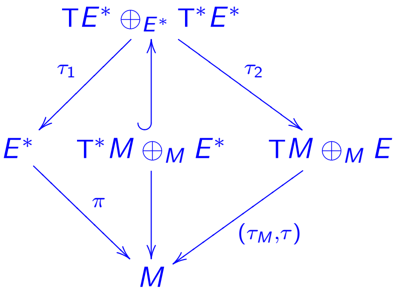

- $Ph_D = \tau_1(D)$ - the phase bundle.
- $Vel_D = \tau_2(D)$ - the velocity bundle (anchor relation).
- $\mathcal{C}_D \subset \mathsf{T}^*M \oplus_M E^*$ - the core bundle of D.

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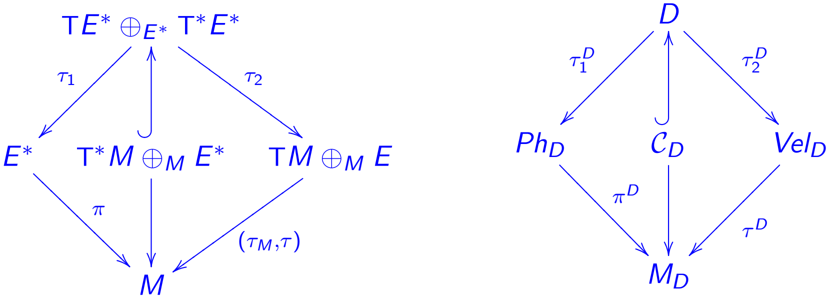

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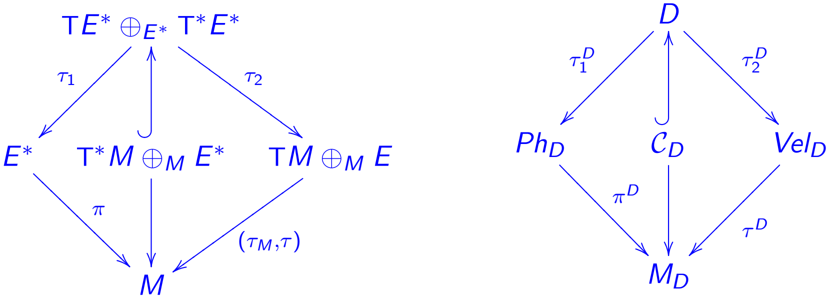

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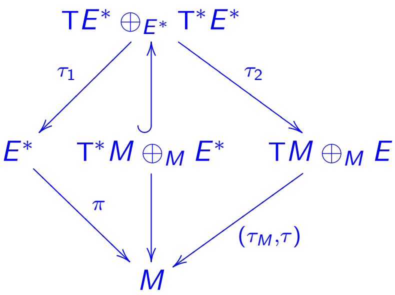

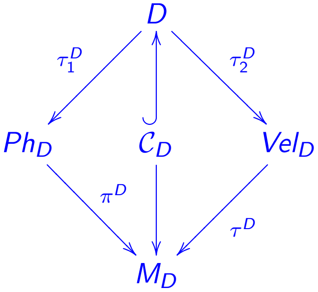

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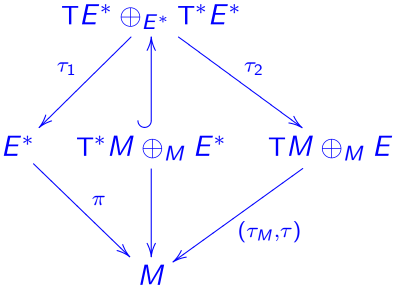

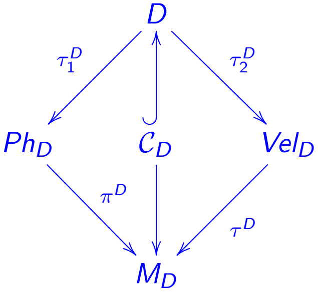

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Dirac algebroids – examples

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$$\Pi = \frac{1}{2}c^k_{ij}(x)\xi_k\partial_{\xi_i}\wedge\partial_{\xi_j} + \rho^b_i(x)\partial_{\xi_i}\wedge\partial_{x^b}, \quad c^k_{ij}(x) = -c^k_{ji}(x)\,,$$

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  where A_0 is the affine subbundle in $\mathcal{T}\mathbb{R}$ defined by the constraint $\dot{x}_0 = 1$ (in the natural coordinates $(x_0, \dot{x}_0, p_0)$), is an affine Dirac algebroid on $\widetilde{\mathsf{T}}M$.

Dirac algebroids

This description of the core of a Dirac algebroid in general gives the following.

Theorem

The core bundle of a Dirac algebroid D is the annihilator subbundle $Vel_D^\perp \subset \mathsf{T}^*M \oplus_M E^*$ of the anchor relation Vel_D: $\mathcal{C}_D = Vel_D^\perp$.

In addition, for Dirac-Lie algebroids we have the following.

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If D is a Dirac-Lie algebroid, then it induces a Lie algebroid structure on the bundle Vel_D.

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In the case of D_Π, it is the Lie algebroid structure on $E \simeq Vel_{D_\Pi}$.
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Theorem

One can find local coordinates $(x, \widehat{x}, \xi, \widehat{\xi}, \eta, \widehat{\eta}, \zeta, \widehat{\zeta})$ *in* $\mathcal{T}E^*$ *such that*

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Tulczyjew Triple

M - positions,
$\mathrm{T}M$ - (kinematic) configurations,
$L : \mathrm{T}M \to \mathbb{R}$ - Lagrangian
T^*M - phase space

$$\mathcal{D} = \alpha_M^{-1}(\mathrm{d}L(\mathrm{T}M)))$$

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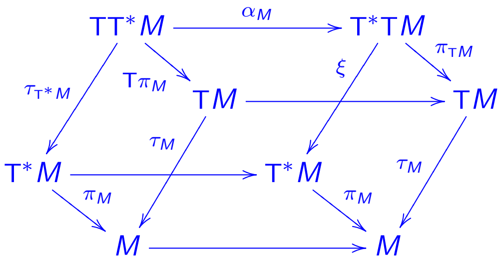

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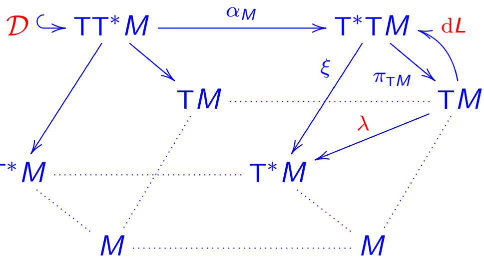

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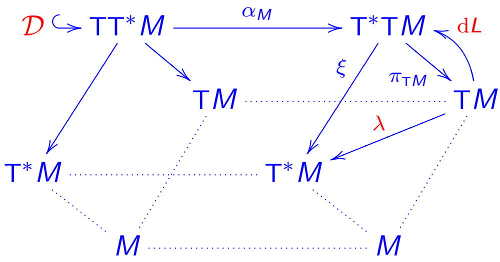

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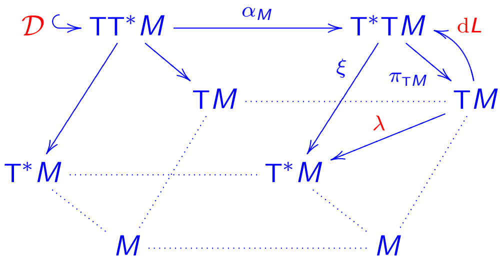

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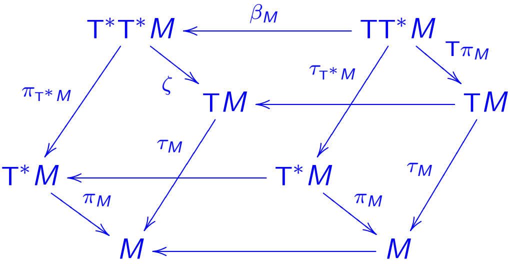

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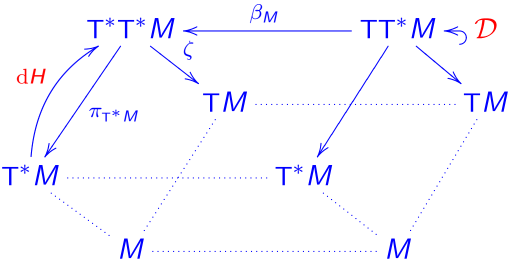

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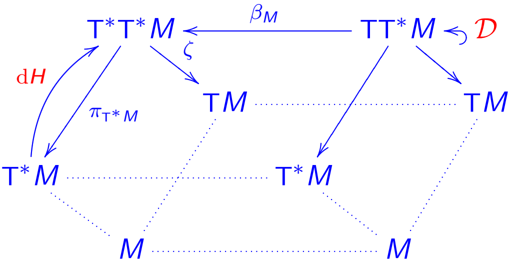

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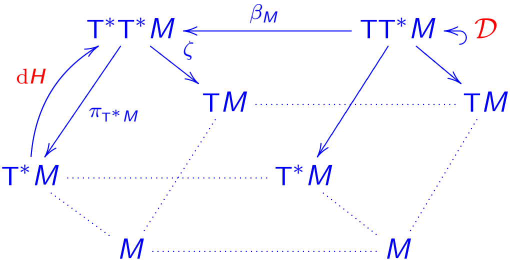

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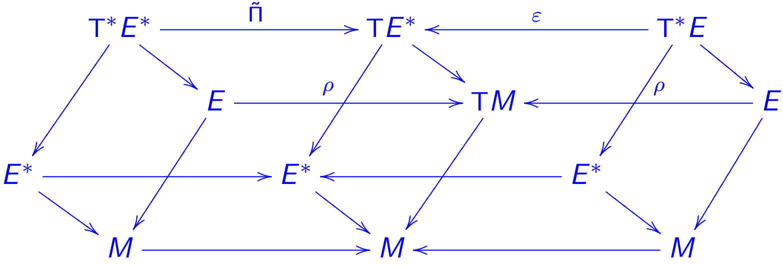
Algebroid setting, no constraints



$H : E^* \longrightarrow \mathbb{R}$ $\qquad$ $\mathcal{D} = \varepsilon(\mathrm{d}L(E))$ $\qquad$ $L : E \longrightarrow \mathbb{R}$

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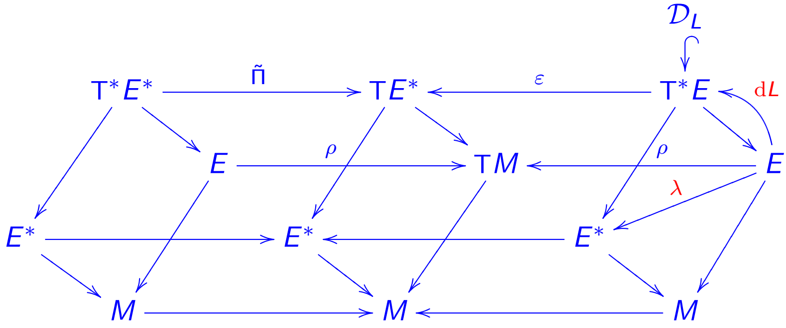

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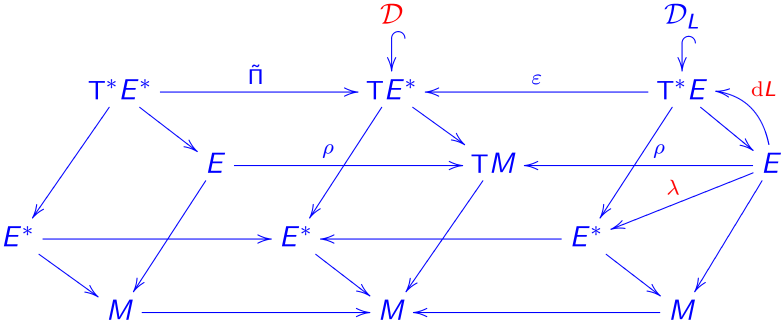

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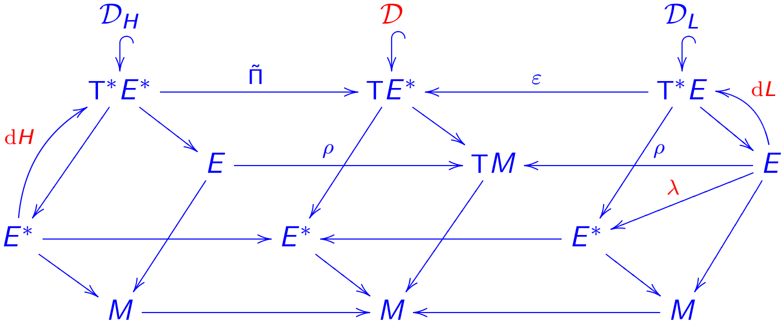

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$$D \subset \mathsf{T}^*E^* \oplus_{E^*} \mathsf{T}E^* \subset \mathsf{T}^*E^* \times \mathsf{T}E^*$$

D can be treated as a relation

$$\beta_D : \mathsf{T}^*E^* \longrightarrow\!\!\!\!\triangleright\, \mathsf{T}E^*.$$

Composing with $\mathcal{R}_E : \mathsf{T}^*E \to \mathsf{T}^*E^*$ we get another relation

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The diagram is commutative in the sense of relations

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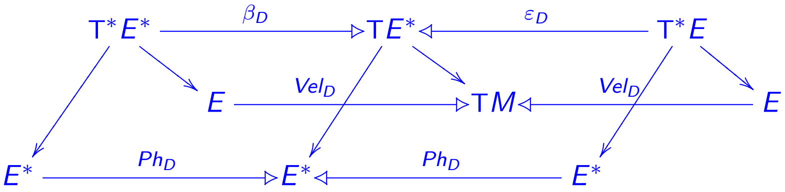

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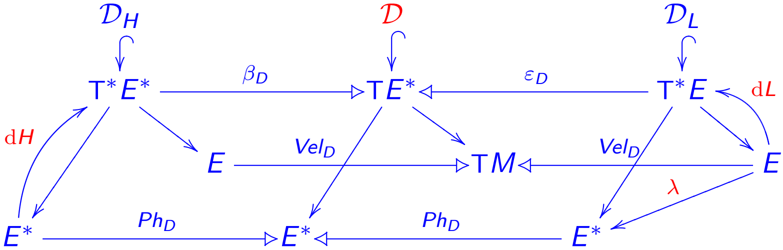

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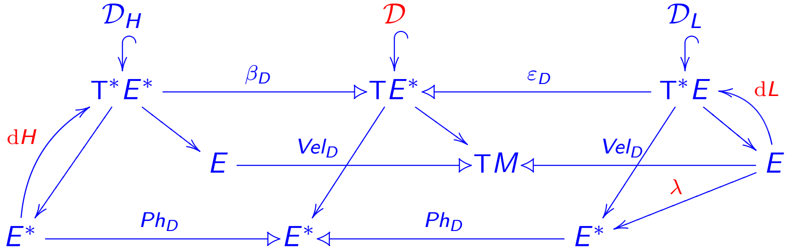

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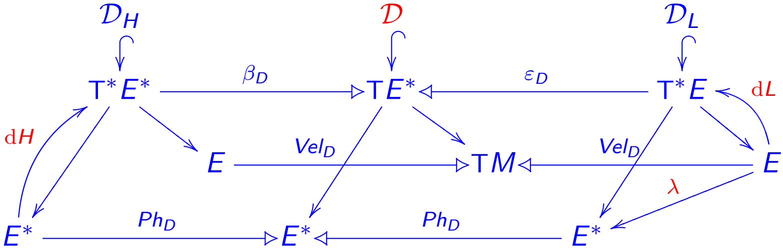

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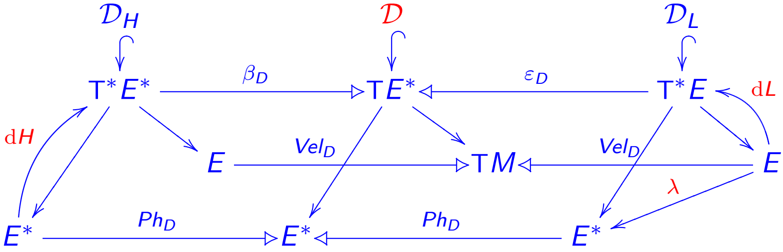

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How to deal with (nonholonomic) constraints?

Remember that constraints should be given not only for kinematic configurations but also for virtual displacements.

There are two possibilities to describe a system with constraints:

1. Keep the structure of the triple unchanged and modify the way of generating $\mathcal{D}_L$ out of L.
2. Keep the way of generating unchanged and modify the structure (Dirac algebroid) of the triple.

The latter is our choice.

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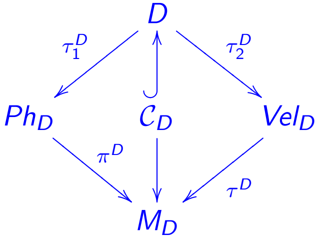
Dirac algebroid induced by constraints

Initial data: Dirac algebroid D on E and a vector subbundle V of Vel_D.



- $V \subset Vel_D \subset \mathsf{T}M \oplus_M E$
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- $V^\perp \subset \mathsf{T}^*M \oplus_M E^*$, $V^\perp \supset \mathcal{C}_D$
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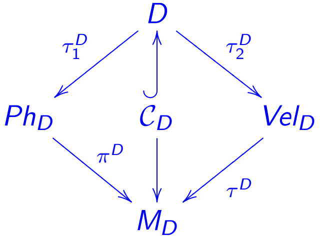

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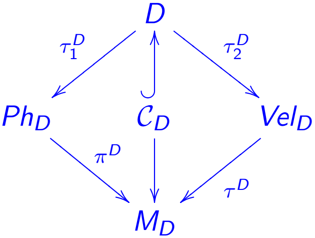

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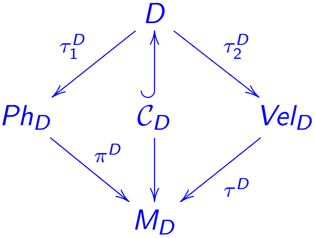

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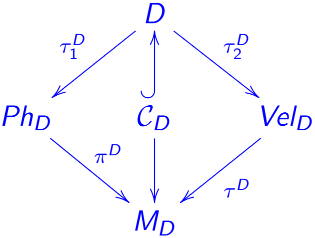

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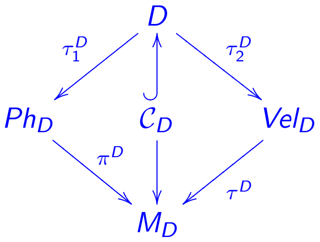

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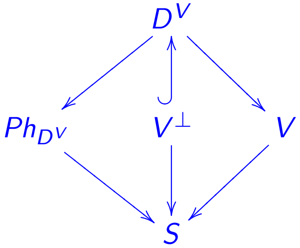

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Mechanics on Dirac algebroids

Let

$$D=\{(x,\widehat{x},\xi,\widehat{\xi},\eta,\widehat{\eta},\zeta,\widehat{\zeta}):\ \ \widehat{x}=0,\ \widehat{\xi}=0,\ \widehat{\eta}=0,\ \zeta_k=c^i_{jk}(x)\eta^j\xi_i\}$$

be a Dirac structure in $\mathcal{T}E^*$ and $L:E\to\mathbb{R}$ be a Lagrangian.

Using standard local coordinates $(x,\widehat{x},\xi,\widehat{\xi},\eta,\widehat{\eta},\zeta,\widehat{\zeta})$, we can write $\xi=\frac{\partial L}{\partial y}(x,y)$ and $p=-\frac{\partial L}{\partial x}(x,y)$.

The (implicit) Euler-Lagrange equations read as

$$\widehat{x}=0\,,\quad \widehat{\xi}\left(x,\frac{\partial L}{\partial y}(x,y)\right)=0\,,\quad \widehat{\eta}(x,\dot{x},y)=0\,,$$

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$$\frac{\mathrm{d}\xi_a}{\mathrm{d}t}=-\frac{\partial H}{\partial x^a}(x,\xi)\,,\quad \frac{\mathrm{d}x^b}{\mathrm{d}t}=\frac{\partial H}{\partial \xi_b}(x,\xi)$$

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Mechanics on presymplectic manifolds

Consider the Dirac algebroid D_ω associated with a linear 2-form ω on E^*,

$$\omega = \frac{1}{2}c^k_{ab}(x)\xi_k \mathrm{d}x^a \wedge \mathrm{d}x^b + \rho^i_b(x)\mathrm{d}\xi_i \wedge \mathrm{d}x^b, \quad c^k_{ab}(x) = -c^k_{ba}(x)\,.$$

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In the case of a regular presymplectic form of rank r,

$$\omega = \sum_{a\le r} \mathrm{d}p_a \wedge \mathrm{d}x^a\,,$$

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$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^a}(x,\dot{x})\right) = -\frac{\partial L}{\partial x^a}(x,\dot{x})\,, \quad a \leq r\,.$$

Nonautonomous and constraint mechanics on algebroids

- For the affine Dirac algebroid $\widetilde{D}_M$ on $\mathsf{T}M \times \mathbb{R}$ and a time-dependent Lagrangian $L : \mathsf{T}M \times \mathbb{R} \to \mathbb{R}$, we get as the Euler-Lagrange equations the nonautonomous equations

$$\frac{\mathrm{d}x^a}{\mathrm{d}t} = \dot{x}^a, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^a}(t, x, \dot{x})\right) = \frac{\partial L}{\partial x^a}(t, x, \dot{x}).$$

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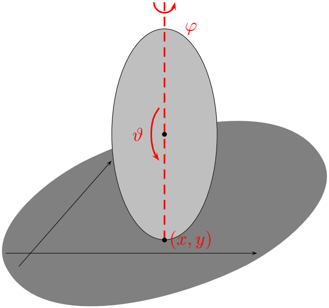
Rolling disc



$N = \mathbb{R}^2 \times S^1 \times S^1 \ni (x, y, \varphi, \vartheta)$

$L : \mathsf{T}N \longrightarrow \mathbb{R}$

$L(v) = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{J_1}{2}\dot{\varphi}^2 + \frac{J_2}{2}\dot{\vartheta}^2$

$\dot{x} = R\dot{\vartheta}\cos\varphi \quad \dot{y} = R\dot{\vartheta}\sin\varphi$

$\Downarrow$ reduction

System on a Lie algebroid

$$\begin{array}{ccc} E = \mathsf{T}S^1 \times \mathbb{R}^3\,, & & (\varphi, \dot{\varphi}, \dot{x}, \dot{y}, \dot{\vartheta}) \\ \downarrow \tau & & \downarrow \\ M = S^1 & & (\varphi) \end{array}$$

$\rho : E \ni (\varphi, \dot{\varphi}, \dot{x}, \dot{y}, \dot{\vartheta}) \mapsto (\varphi, \dot{\varphi}) \in \mathsf{T}S^1$

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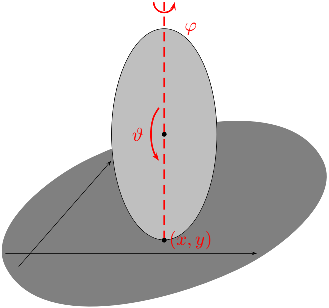

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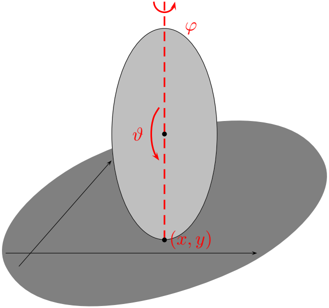

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Rolling disc

New coordinates (φ, y^i) in E associated to global sections

$$e_1 = \partial_\varphi, \quad e_2 = \partial_\vartheta + R\cos\varphi\partial_x + R\cos\varphi\partial_y, \quad e_3 = \partial_x, \quad e_4 = \partial_y$$

$$[e_1, e_2] = R\cos\varphi e_4 - R\sin\varphi e_3$$

$D \subset \mathsf{T}E^* \oplus_{E^*} \mathsf{T}^*E^*$

$(\varphi, \xi_i, \dot\varphi, \dot\xi_j, p, y^k)$:

$\dot\varphi = y^1$

$\dot\xi_1 = y^2(R\xi_3\sin\varphi - R\xi_4\cos\varphi) - p$

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$\dot\xi_3 = \dot\xi_4 = 0$

$\{(\varphi, y):\ y^3 = y^4 = 0\} \subset E$

$Vel_D \supset V = Vel_{DV}$

$\{(\varphi, y, \dot\varphi):\ y^3 = y^4 = 0, y^1 = \dot\varphi\}$

$C_D \subset V^\perp$

$\{(\varphi, p, \dot\xi_i): p + \dot\xi_1 = 0,\ \dot\xi_2 = 0$

$\dot\xi_3,\ \dot\xi_4$ – arbitrary $\}$

$Ph_{DV} = E^*$

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The dynamics $\mathcal{D} \subset \mathsf{T}E^*$ (admits a Hamiltonian !)

$$\mathcal{D} = \quad \{(\varphi, \xi_i, \dot{\varphi}, \dot{\xi}_j) : \xi_3 = \frac{mR}{mR^2 + J_2}\xi_2 \cos\varphi\,, \dot{\varphi} = \frac{1}{J_1}\xi_1\,,$$
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Note that $\dot{\xi}_3\,, \dot{\xi}_4$ in $\mathcal{D}$ are arbitrary, but ξ_3, ξ_4 are determined by integrability conditions.

The constrained Euler-Lagrange equations are

$$y^3 = y^4 = 0\,,\ \frac{\mathrm{d}\varphi}{\mathrm{d}t} = y^1\,, (mR^2 + J_2)\frac{\mathrm{d}y^2}{\mathrm{d}t} = 0\,,\ J_1\frac{\mathrm{d}y^1}{\mathrm{d}t} = 0\,.$$

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$$y^3 = y^4 = 0\,,\ \frac{\mathrm{d}\varphi}{\mathrm{d}t} = y^1\,, (mR^2 + J_2)\frac{\mathrm{d}y^2}{\mathrm{d}t} = 0\,,\ J_1\frac{\mathrm{d}y^1}{\mathrm{d}t} = 0\,.$$

that can be rewritten as

$$\dot{x}^1 = R\,\dot{\theta}\cos\varphi\,,\ \dot{x}^2 = R\,\dot{\theta}\sin\varphi\,,\ \ddot{\varphi} = 0\,,\ \ddot{\theta} = 0\,.$$

Conclusions

Canonical symplectic forms on cotangent bundles, skew-algebroid structures, Dirac algebroids induced by constraints, etc., are all special examples of a Dirac algebroid. We can therefore describe all main types of mechanical systems using this single geometric structure!

THANK YOU!

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Happy Birthday to Włodek Tulczyjew!