On the universal covering groups of C^r-diffeomorphism groups

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Introduction

- Group G is perfect iff $G = [G, G]$, where $[G, G]$ is generated by all elements of the form $fgf^{-1}g^{-1}$, $f, g \in G$. In other words $H_1(G) = G/[G, G] = 0$.
- G is simple if it has no proper normal subgroups.
- Every nonabelian, simple group is perfect.
- Inversely, Epstein proved that perfectness implies simplicity for a large class of groups of homeomorphisms with compact supports, if the transitivity condition holds.

Introduction

We denote

- M - smooth, connected manifold, where $\dim M = n$, $n \geq 1$
- $r = 1, \ldots, \infty$
- $\mathrm{Diff}^r_c(M)$ - group of C^r-diffeomorphisms on M with compact supports isotopic to Id through compactly supported C^r-isotopies
- $\widetilde{\mathrm{Diff}^r_c(M)}$ - universal covering group of $\mathrm{Diff}^r_c(M)$

Theorem (Mather 1974, 1975, Epstein 1984)

The group $\mathrm{Diff}^r_c(M)$ is perfect and simple for $r \neq n+1$ or $r = \infty$.

For $r = n+1$ the problem is open. (Mather 1984, 1985)

Theorem (Thurston 1974)

The group $\mathrm{Diff}^\infty_c(M)$ is perfect and simple. Moreover the universal covering group $\widetilde{\mathrm{Diff}^\infty_c(M)}$ is perfect.

We get the following

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Let M be n-dimensional, connected manifold. Then $\widetilde{\mathrm{Diff}^r_c(M)}$ is perfect for $r \neq n+1$, r finite.

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Groups of paths

- G - topological group
- $\mathcal{P}G = \{\gamma : [0,1] \to G \text{ - continuous}, \gamma_0 = e\}$
  - group of continuous paths in G

  With pointwise multiplication it is topological group in the compact-open topology.
- $\widetilde{G} = \mathcal{P}G/\sim$ - universal covering group of G, where $\sim$ is the relation of the homotopy relative to the endpoints.
- $\langle\gamma\rangle_\sim \in \widetilde{G}$ - equivalence class of $\gamma \in \mathcal{P}G$

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- Let us take
  $\mathcal{P}^*G = \{\gamma \in \mathcal{P}G \,:\, \gamma_t = e,\ t \in [0, \frac{1}{2}]\}$ - special paths in G,
  $\mathcal{P}^\square G = \{\gamma \in \mathcal{P}G \,:\, \gamma_t = \gamma_1,\ t \in [\frac{1}{2}, 1]\}$.
- For $\gamma \in \mathcal{P}G$ we set $\gamma^* \in \mathcal{P}^*G$ and $\gamma^\square \in \mathcal{P}^\square G$ by

$$\gamma^*_t = \begin{cases} e & t \in [0, \frac{1}{2}] \\ \gamma_{2t-1} & t \in [\frac{1}{2}, 1] \end{cases} \qquad \gamma^\square_t = \begin{cases} \gamma_{2t} & t \in [0, \frac{1}{2}] \\ \gamma_1 & t \in [\frac{1}{2}, 1] \end{cases}$$

Lemma

For every $\gamma \in \mathcal{P}G$ there is $\gamma \sim \gamma^*$ and $\gamma \sim \gamma^\square$.

- $\mathcal{P}^*G$ is preserved by conjugation, i.e. $\mathrm{conj}_f(\mathcal{P}^*G) \subset \mathcal{P}^*G$, where $\mathrm{conj}_f(g) = fgf^{-1}$, $f, g \in \mathcal{P}G$.

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Moduli of continuity

Definition

A function $\alpha : [0, \infty) \to \mathbb{R}$ is a modulus of continuity iff α is continuous, strictly increasing, $\alpha(0) = 0$ and $\alpha(st) \leq s\alpha(t)$ for $t \geq 0$, $s \geq 1$.

Definition

Let X, Y be metric spaces. A mapping $f : X \to Y$ is α-continuous iff there exist constants $C > 0$ and $\varepsilon > 0$ such that

$$d(x, y) \leq \varepsilon \quad \Rightarrow \quad d(f(x), f(y)) \leq C\alpha(d(x, y)) \quad x, y \in X.$$

Moreover, f is locally α-continuous if for every $x \in X$ there exists a neighbourhood U of x such that $f|_U$ is α-continuous.

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For a modulus of continuity α

- $\mathrm{Diff}^{r,\alpha}_c(M)$ - the group of $C^{r,\alpha}$-diffeomorphisms isotopic to Id through compactly supported $C^{r,\alpha}$-isotopies
- $\mathrm{Diff}^{r,\alpha}_K(M) \subset \mathrm{Diff}^{r,\alpha}_c(M)$ - subgroup of diffeomorphisms with supports in K, where $K \subset M$ is closed
- $\mathcal{P}\,\mathrm{Diff}^{r,\alpha}_c(M)$ - space of paths in $\mathrm{Diff}^{r,\alpha}_c(M)$
- $\mathcal{P}^*\,\mathrm{Diff}^{r,\alpha}_c(M)$ - space of special paths in $\mathrm{Diff}^{r,\alpha}_c(M)$

- For $f = \{f_t\}_{t\in[0,1]} \in \mathcal{P}\,\mathrm{Diff}^{r,\alpha}_c(M)$ we define

$$\mathrm{supp}(f) = \bigcup_{t\in[0,1]} \mathrm{supp}(f_t)$$

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Theorem (Lech, Michalik 2011)

Let M be n-dimensional, connected manifold. Then the group $\widetilde{\mathrm{Diff}_c^r(M)}$ is perfect, whenever $r \neq n+1$, r finite.

- **Fragmentation property:**
  Let $\{U_j\}_{j\in J}$ be a covering of M by open balls. Then for every $f \in \mathcal{P}\,\mathrm{Diff}_c^r(M)$ there there exist $f_1, \dots, f_l \in \mathcal{P}\,\mathrm{Diff}_c^r(M)$ such that $f = f_1 \dots f_l$ and $\mathrm{supp}(f_i) \subset U_{j(i)}$.

Hence it sufficies to show the proof for $M = \mathbb{R}^n$.

- Moreover the following equality holds

$$\mathcal{P}\,\mathrm{Diff}_c^r(\mathbb{R}^n) = \bigcup_\alpha \mathcal{P}\,\mathrm{Diff}_c^{r;\alpha}(\mathbb{R}^n),$$

  where the sum is over all moduli of continuity α such that $\alpha(st) \leq \sqrt{s}\alpha(t)$ for every $t \geq 0$, $s \geq 1$.

We will prove that $\widetilde{\mathrm{Diff}_c^{r;\alpha}(\mathbb{R}^n)}$ is perfect for every α as above.

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Seminorms

- Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $r \geq 0$.
  We define seminorms

$$\mu_r(f) = \sup_{x \in \mathbb{R}^n} \| D^r f(x) - \mathrm{Id} \|$$

$$\mu_{r,\alpha}(f) = \sup_{x \neq y} \frac{\|D^r f(x) - D^r f(y)\|}{\alpha(|x - y|)}$$

- For a path $f = \{f_t\}_{t \in [0,1]} \in \mathcal{P}\,\mathrm{Diff}^{r,\alpha}(\mathbb{R}^n)$ we set

$$\mu_r^*(f) = \sup_{t \in [0,1]} \mu_r(f_t), \quad \mu_{r,\alpha}^*(f) = \sup_{t \in [0,1]} \mu_{r,\alpha}(f_t)$$

Lemma

Let $K \subset \mathbb{R}^n$ be closed and such that $R_K = \sup\{\operatorname{dist}(x, \overline{\mathbb{R}^n \setminus K}) \, : \, x \in \mathbb{R}^n\} < \infty$.

1. *There exist $\delta_1 > 0$ and $C_1 > 0$ depending on n, r, α and R_K such that*
$$\mu^*_{r,\alpha}(fg) \leq \mu^*_{r.\alpha}(f) + \mu^*_{r,\alpha}(g) + C_1\mu^*_{r,\alpha}(f)\mu^*_{r,\alpha}(g)$$
for all $f, g \in \mathcal{P}\operatorname{Diff}^{r,\alpha}(\mathbb{R}^n)$ with $\mu^*_{r,\alpha}(f), \mu^*_{r,\alpha}(g) \leq \delta_1$ and $D^1(f_t - \operatorname{Id}) = D^1(g_t - \operatorname{Id}) = 0$ on $\mathbb{R}^n \setminus K$, $t \in [0, 1]$.
2. *There exists $\delta_2 > 0$ depending on n, r, α and R_K such that*
$$\mu^*_{r,\alpha}(f^{-1}) \leq 2\mu^*_{r,\alpha}(f)$$
for every $f \in \mathcal{P}\operatorname{Diff}^{r,\alpha}(\mathbb{R}^n)$ with $\mu^*_{r,\alpha}(f) \leq \delta_2$ and $D^1(f_t - \operatorname{Id}) = 0$ on $\mathbb{R}^n \setminus K$, $t \in [0, 1]$.

Let $A \geq 1$, $A \in \mathbb{N}$, $1 \leq r \leq n$, and let α be a modulus of continuity.

- We define

$$K = [-2A, 2A]^n$$
$$K_i = [-2, 2]^i \times [-2A, 2A]^{n-i}, \quad i = 0, \dots, n$$

Then

$$K_n = [-2, 2]^n \subset K_{n-1} \subset \dots \subset K_0 = [-2A, 2A]^n = K$$

- Fix $\widetilde{\zeta_A} \in \mathrm{Diff}_c^\infty(\mathbb{R}^n)$ such that $\widetilde{\zeta_A} = A \cdot \mathrm{Id}$ on K_n.
We define $\zeta_A \in \mathcal{P}\,\mathrm{Diff}_c^\infty(\mathbb{R}^n)$ by the formula

$$(\zeta_A)_t = t(\widetilde{\zeta_A} - \mathrm{Id}) + \mathrm{Id}\,.$$

Then ζ_A is a path from $(\zeta_A)_0 = \mathrm{Id}$ to $(\zeta_A)_1 = \widetilde{\zeta_A}$.

Proof for $r \leq n$

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Proof for $r \leq n$ - Mather's operator

Lemma

There exists U_A a neighbourhood of $\mathrm{Id} \in \mathcal{P}^* \mathrm{Diff}^1_K(\mathbb{R}^n)$ and continuous operators

$$\Psi_{i,A} : U_A \cap \mathcal{P}^* \mathrm{Diff}^{r,\alpha}_{K_i}(\mathbb{R}^n) \to \mathcal{P}^* \mathrm{Diff}^{r,\alpha}_{K_{i-1}}(\mathbb{R}^n), \quad i = 1, \dots, n$$

such that

1. *For every $f \in U_A \cap \mathcal{P}^* \mathrm{Diff}^{r,\alpha}_{K_i}(\mathbb{R}^n)$*
$$[f] = [\Psi_{i,A}(f)] \in H_1(\mathcal{P}\,\mathrm{Diff}^{r,\alpha}_c(\mathbb{R}^n)).$$
2. *There exist $\delta > 0$ depending on r, A, α and $Q \geq 1$ depending on r, α such that*
$$\mu^*_{r,\alpha}(\Psi_{i,A}(f)) \leq \frac{Q}{A}\mu^*_{r,\alpha}(f)$$
whenever $f \in U_A \cap \mathcal{P}^* \mathrm{Diff}^{r,\alpha}_{K_i}(\mathbb{R}^n)$ with $\mu^*_{r,\alpha}(f) \leq \delta$.

Proof for $r \leq n$

Proof for special paths:

- Let $f, g \in \mathcal{P}^* \operatorname{Diff}_K^{r,\alpha}(\mathbb{R}^n)$ be close to Id, $K = [-2A, 2A]^n$.
- We define $g_n, \ldots, g_0 \in \mathcal{P}^* \operatorname{Diff}_K^{r,\alpha}(\mathbb{R}^n)$ by

$$g_n = (\zeta_A^{\square})^{-1} f g \zeta_A^{\square}, \quad g_i = \Psi_{i,A}(g_{i+1}), \quad i = n-1, \ldots, 0.$$

  We get $\operatorname{supp}(g_n) \subset K_n = [-2, 2]^n$ and $\operatorname{supp}(g_i) \subset K_i$.
- Set $\bar{g} = g_0 \in \mathcal{P}^* \operatorname{Diff}_K^{r,\alpha}(\mathbb{R}^n)$.
- For small $\mu^*_{r,\alpha}(f)$, $\mu^*_{r,\alpha}(g)$ there is

$$\mu^*_{r,\alpha}(\bar{g}) \leq C A^{r - \frac{1}{2} - n} (\mu^*_{r,\alpha}(f) + \mu^*_{r,\alpha}(g)).$$

  Hence there are large $A_0 \geq 1$ and small $\varepsilon_0 > 0$ such that

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- Set $\bar{g} = g_0 \in \mathcal{P}^* \operatorname{Diff}_K^{r,\alpha}(\mathbb{R}^n)$.
- For small $\mu_{r,\alpha}^*(f)$, $\mu_{r,\alpha}^*(g)$ there is

$$\mu_{r,\alpha}^*(\bar{g}) \leq CA^{r-\frac{1}{2}-n}(\mu_{r,\alpha}^*(f) + \mu_{r,\alpha}^*(g)).$$

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- We define

$$L = \{h \in \mathrm{Diff}^{r,\alpha}_K(\mathbb{R}^n) \;:\; \mu_{r,\alpha}(h) \leq \varepsilon_0\}$$
$$\mathcal{P}^* L = \{h \in \mathcal{P}^* \mathrm{Diff}^{r,\alpha}_K(\mathbb{R}^n) \;:\; \mu^*_{r,\alpha}(h) \leq \varepsilon_0\}$$
$$\mathcal{P} L = \{h \in \mathcal{P} \mathrm{Diff}^{r,\alpha}_K(\mathbb{R}^n) \;:\; \mu^*_{r,\alpha}(h) \leq \varepsilon_0\}$$

- For fixed $f \in \mathcal{P}^* L$ we get the operator

$$\Phi : \mathcal{P}^* L \ni g \mapsto \bar{g} \in \mathcal{P}^* L.$$

It induces mapping

$$\widetilde{\Phi} : L \ni g_1 \mapsto \bar{g}_1 \in L,$$

where g_1 and $\bar{g}_1$ are ends of paths g and $\bar{g}$.

- From Schauder-Tichonoff fixed-point theorem there is $g_1 \in L$ such that $\widetilde{\Phi}(g_1) = \bar{g}_1 = g_1$ for some paths $g, \bar{g} \in \mathcal{P}^* L$ with $\Phi(g) = \bar{g}$.

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- For fixed $f \in \mathcal{P}^* L$ we get the operator

$$\Phi : \mathcal{P}^* L \ni g \mapsto \bar{g} \in \mathcal{P}^* L.$$

It induces mapping
$$\widetilde{\Phi} : L \ni g_1 \mapsto \bar{g}_1 \in L,$$
where g_1 and $\bar{g}_1$ are ends of paths g and $\bar{g}$.

- From Schauder-Tichonoff fixed-point theorem there is $g_1 \in L$ such that $\widetilde{\Phi}(g_1) = \bar{g}_1 = g_1$ for some paths $g, \bar{g} \in \mathcal{P}^* L$ with $\Phi(g) = \bar{g}$.

Proof for $r \leq n$

- We define

$$L = \{h \in \mathrm{Diff}^{r,\alpha}_K(\mathbb{R}^n) \ : \ \mu_{r,\alpha}(h) \leq \varepsilon_0\}$$
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Proof for $r \leq n$

- The group $\mathrm{Diff}^{r,\alpha}_K(\mathbb{R}^n)$ is locally contractible. Hence the concatenation $\bar{g}^{-1} * g$ is a loop homotopic with $\mathrm{Id} \in \mathcal{P}L$ for sufficiently small $\varepsilon_0 > 0$.
- It follows that $\langle g \rangle_\sim = \langle \bar{g} \rangle_\sim$ in $\mathcal{P}L/\sim$.
- From properties of Mather's operator we get

$$[fg] = [g_n] = \ldots = [g_0] = [\bar{g}] \in H_1(\mathcal{P}\,\mathrm{Diff}^{r,\alpha}_c(\mathbb{R}^n)).$$

- Then $[\langle fg \rangle_\sim] = [\langle \bar{g} \rangle_\sim]$ in $H_1(\widetilde{\mathrm{Diff}^{r,\alpha}_c(\mathbb{R}^n)})$ and

$$[\langle f \rangle_\sim] = [\mathrm{Id}] \in H_1(\widetilde{\mathrm{Diff}^{r,\alpha}_c(\mathbb{R}^n)})$$

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Proof for $r \leq n$

For an arbitrary path $f \in \mathcal{P}L$:

- We take special path $f^* \in \mathcal{P}^*L$.
- As $f \sim f^*$ then $\langle f \rangle_\sim = \langle f^* \rangle_\sim \in \widetilde{\mathrm{Diff}_c^{r,\alpha}(\mathbb{R}^n)}$.
  It means that
$$[\langle f \rangle_\sim] = [\langle f^* \rangle_\sim] = [\mathrm{Id}] \in H_1(\widetilde{\mathrm{Diff}_c^{r,\alpha}(\mathbb{R}^n)}).$$

Lemma

There exist U_A *a neighbourhood of* Id *in* $\mathcal{P}^*\operatorname{Diff}^1_{K_0}(\mathbb{R}^n)$ *depending on* A *and continuous operators*

$$\Psi_{i,A} : U_A \cap \mathcal{P}^*\operatorname{Diff}^{r,\alpha}_{K_{i-1}}(\mathbb{R}^n) \to \mathcal{P}^*\operatorname{Diff}^{r,\alpha}_{K_i}(\mathbb{R}^n), \quad i = 1, \dots, n$$

such that

1. *For* $f \in U_A \cap \mathcal{P}^*\operatorname{Diff}^{r,\alpha}_{K_{i-1}}(\mathbb{R}^n)$
$$[f] = [\Psi_{i,A}(f)] \in H_1(\mathcal{P}\operatorname{Diff}^{r,\alpha}_c(\mathbb{R}^n)).$$
2. *There exist* $\delta > 0$ *depending on* r, A, α *and* $Q \geq 1$ *depending on* r, α *such that*
$$\mu^*_{r,\alpha}(\Psi_{i,A}(f)) \leq QA\mu^*_{r,\alpha}(f)$$
for every $f \in U_A \cap \mathcal{P}^*\operatorname{Diff}^{r,\alpha}_{K_{i-1}}(\mathbb{R}^n)$ *with* $\mu^*_{r,\alpha}(f) \leq \delta$.

The case $r > n+1$

- We take $A \geq 1$ and $\zeta_A \in \mathcal{P}\,\mathrm{Diff}_c^\infty(\mathbb{R}^n)$ as above. Now

$$K = [-2,2]^n = K_n \subset \ldots \subset K_0 = [-2A, 2A]^n.$$

- For $f, g \in \mathcal{P}^*\,\mathrm{Diff}_K^{r,\alpha}(\mathbb{R}^n)$ close to Id we denote

$$g_0 = \zeta_A^\square fg(\zeta_A^\square)^{-1}, \quad g_i = \Psi_{i,A}(g_{i-1}), \quad i = 1, \ldots, n.$$

  Set $\bar{g} = g_n$.

- We have

$$\mu^*_{r,\alpha}(\bar{g}) \leq CA^{1-r+n}(\mu^*_{r,\alpha}(f) + \mu^*_{r,\alpha}(g)).$$

  Hence for small $\varepsilon_0 > 0$ and large $A_0 > 0$

$$\mu^*_{r,\alpha}(f), \mu^*_{r,\alpha}(g) \leq \varepsilon_0 \quad \Rightarrow \quad \mu^*_{r,\alpha}(\bar{g}) \leq \varepsilon_0.$$

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$$\Phi : \mathcal{P}^* L \ni g \mapsto \bar{g} \in \mathcal{P}^* L, \quad \widetilde{\Phi} : L \ni g_1 \mapsto \bar{g}_1 \in L.$$

- From fixed-point theorem there exists $g \in \mathcal{P}^* L$ with $\Phi(g) = \bar{g}$ such that $\widetilde{\Phi}(g_1) = g_1$, where g_1 is the end of path g.
- The concatenation $\bar{g}^{-1} * g$ is a loop homotopic with $\mathrm{Id} \in \mathcal{P} L$ for sufficiently small $\varepsilon_0 > 0$ and hence $\langle g \rangle_\sim = \langle \bar{g} \rangle_\sim$.
- We obtain

$$[fg] = [\bar{g}] \in H_1(\mathcal{P}\,\mathrm{Diff}_c^{r,\alpha}(\mathbb{R}^n))$$

and then

$$[\langle f \rangle_\sim] = [\mathrm{Id}] \in H_1(\widetilde{\mathrm{Diff}_c^{r,\alpha}(\mathbb{R}^n)}).$$

for every $f \in \mathcal{P}^* L$.