Uniform perfectness and uniform simplicity of certain homeomorphism groups

Ilona Michalik

(joint work with Tomasz Rybicki)

AGH University of Science and Technology
Faculty of Applied Mathematics

Geometry of Manifolds and Mathematical Physics
Krakow 27th June 2011 - 1st July 2011
to celebrate the 80th birthday of Wlodzimierz Tulczyjew

Ling's theorem

- Let $\mathcal{U}$ be an open cover of X. A group of homeomorphisms G of a space X is called $\mathcal{U}$-**factorizable** if for every $g \in G$ there are $g_1, \ldots, g_r \in G$ with $g = g_1 \ldots g_r$ and such that $\mathrm{supp}(g_i) \subset U_i$, $i = 1, \ldots, r$, for some $U_1, \ldots, U_r \in \mathcal{U}$. G is called **factorizable** if for every open cover $\mathcal{U}$ of X it is $\mathcal{U}$-factorizable.
- G is said to be **non-fixing** if $G(x) \neq \{x\}$ for every $x \in X$, where $G(x) := \{g(x) | g \in G\}$ is the orbit of G at x.
- Given a group G, denote by $[f, g] = fgf^{-1}g^{-1}$ the commutator of $f, g \in G$, and by $[G, G]$ the commutator subgroup.

- Let $\mathcal{U}$ be an open cover of X. A group of homeomorphisms G of a space X is called $\mathcal{U}$-**factorizable** if for every $g \in G$ there are $g_1, \ldots, g_r \in G$ with $g = g_1 \ldots g_r$ and such that $\mathrm{supp}(g_i) \subset U_i$, $i = 1, \ldots, r$, for some $U_1, \ldots, U_r \in \mathcal{U}$. G is called **factorizable** if for every open cover $\mathcal{U}$ of X it is $\mathcal{U}$-factorizable.
- G is said to be **non-fixing** if $G(x) \neq \{x\}$ for every $x \in X$, where $G(x) := \{g(x) | g \in G\}$ is the orbit of G at x.
- Given a group G, denote by $[f, g] = fgf^{-1}g^{-1}$ the commutator of $f, g \in G$, and by $[G, G]$ the commutator subgroup.

Ling's theorem [3]

Let X be a paracompact topological space and let G be a factorizable non-fixing group of homeomorphisms of X. Then the commutator subgroup $[G, G]$ is perfect, that is $[[G, G], [G, G]] = [G, G]$.

- A group G is called **uniformly perfect** ([1]) if G is perfect (i.e. $G = [G, G]$) and there exists a positive integer r such that any element of G can be expressed as a product of at most r commutators of elements of G.
- For $g \in [G, G]$, $g \neq e$, the least r such that g is a product of r commutators is called the **commutator length** of g and is denoted by $\mathrm{cl}_G(g)$. By definition we put $\mathrm{cl}_G(e) = 0$.
- A **conjugation-invariant norm** (or *norm* for short) on group G is a function $\nu : G \to [0, \infty)$ which satisfies the following conditions. For any $g, h \in G$
  1. $\nu(g) > 0$ if and only if $g \neq e$;
  2. $\nu(g^{-1}) = \nu(g)$;
  3. $\nu(gh) \leq \nu(g) + \nu(h)$;
  4. $\nu(hgh^{-1}) = \nu(g)$.
- A group is called **bounded** if it is bounded with respect to any bi-invariant metric. It is easily seen that G is bounded if and only if any conjugation-invariant norm on G is bounded.

- The commutator length cl_G is a conjugation-invariant norm on $[G, G]$. In particular, if G is a perfect group then cl_G is a conjugation-invariant norm on G. For any perfect group G denote by cld_G the **commutator length diameter** of G, i.e. $\mathrm{cld}_G := \sup_{g\in G} \mathrm{cl}_G(g)$. Then G is uniformly perfect iff $\mathrm{cld}_G < \infty$.

- Let $\mathcal{U}$ be an open cover of X and $G \leq \mathcal{H}(X)$.
  1. G is called 1-**non-fixing** if for any $x \in X$ there are $f, g \in G$ such that $([f, g])(x) \neq x$.
  2. G is said to be $\mathcal{U}$-**moving** if for every $U \in \mathcal{U}$ there is $g \in G$ such that $g(U) \cap U = \emptyset$.
  3. G is said to be **strongly** $\mathcal{U}$-**moving** if for every $U, V \in \mathcal{U}$ there is $g \in G$ such that $g(U) \cap (U \cup V) = \emptyset$.
  4. G is called **locally moving** if for any open set $U \subset X$ and $x \in U$ there is $g \in G_U$ such that $g(x) \neq x$.

Assume that $G \leq \mathcal{H}(X)$ is $\mathcal{U}$-factorizable, and that $\mathcal{U}$ is a G-invariant open cover of X ($g(U) \in \mathcal{U}$ for all $g \in G$ and $U \in \mathcal{U}$). Then we may introduce the following conjugation-invariant norm $\mathrm{frag}^{\mathcal{U}}$ on G. Namely, for $g \in G$, $g \neq \mathrm{id}$, we define $\mathrm{frag}^{\mathcal{U}}(g)$ to be the least integer $\rho > 0$ such that $g = g_1 \dots g_\rho$ with $\mathrm{supp}(g_i) \subset U_i$ for some $U_i \in \mathcal{U}$, where $i = 1, \dots, \rho$. By definition $\mathrm{frag}^{\mathcal{U}}(\mathrm{id}) = 0$.
Define $\mathrm{fragd}^{\mathcal{U}}_G := \sup_{g\in G} \mathrm{frag}^{\mathcal{U}}(g)$, the diameter of G in $\mathrm{frag}^{\mathcal{U}}$. Consequently, $\mathrm{frag}^{\mathcal{U}}$ is bounded iff $\mathrm{fragd}^{\mathcal{U}}_G < \infty$.

Uniform perfectness of [G,G]

Theorem 1

Let X be a paracompact topological space and let G be a factorizable non-fixing group of homeomorphisms of X. Assume that cl_G is bounded on $[G, G]$ and that G is bounded with respect to all fragmentation norms $\mathrm{frag}^{\mathcal{U}}$, where $\mathcal{U}$ runs over all open covers of X. Then the commutator subgroup $[G, G]$ is uniformly perfect.

Uniform perfectness of [G,G]

Theorem 1

Let X be a paracompact topological space and let G be a factorizable non-fixing group of homeomorphisms of X. Assume that cl_G is bounded on $[G, G]$ and that G is bounded with respect to all fragmentation norms $\mathrm{frag}^{\mathcal{U}}$, where $\mathcal{U}$ runs over all open covers of X. Then the commutator subgroup $[G, G]$ is uniformly perfect.

Corollary 1

Let X be a paracompact space and let $G \leq \mathcal{H}(X)$ be a bounded, factorizable and non-fixing group. Then the commutator subgroup $[G, G]$ is uniformly perfect.

Uniform perfectness of [G,G]

Proposition

Let X be paracompact and let $G \leq \mathcal{H}(X)$.

1. If G is non-fixing and factorizable then G is locally moving.
2. If G is locally moving then so is $[G, G]$.
3. If G is non-fixing and factorizable then $[G, G]$ is 1-non-fixing.

Proposition

Let X be paracompact and let $G \leq \mathcal{H}(X)$.

1. If G is non-fixing and factorizable then G is locally moving.
2. If G is locally moving then so is $[G, G]$.
3. If G is non-fixing and factorizable then $[G, G]$ is 1-non-fixing.

Lemma

If X is a paracompact space and $\mathcal{U}$ is an open cover of X, then there exists an open cover $\mathcal{V}$ star finer than $\mathcal{U}$, that is for all $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $\mathrm{star}^{\mathcal{V}}(V) \subset U$. Here $\mathrm{star}^{\mathcal{V}}(V) := \bigcup\{V' \in \mathcal{V} : \ V' \cap V \neq \emptyset\}$. In particular, for all $V_1, V_2 \in \mathcal{V}$ with $V_1 \cap V_2 \neq \emptyset$ there is $U \in \mathcal{U}$ such that $V_1 \cup V_2 \subset U$.

If $\mathcal{V}$ and $\mathcal{U}$ are as in Lemma then we will write $\mathcal{V} \prec \mathcal{U}$.

Uniform perfectness of [G,G]

Theorem 2

Let X be a paracompact topological space, let $G \leq \mathcal{H}(X)$ with cl_G bounded (as the norm on $[G, G]$) and let $\mathcal{U}$ be a G-invariant open cover of X such that

1. G is strongly $\mathcal{U}$-moving, and
2. there is an open cover $\mathcal{V}$ satisfying $\mathcal{V} \prec \mathcal{U}$ such that G is $\mathcal{V}$-factorizable and G is bounded with respect to the fragmentation norm $\mathrm{frag}^{\mathcal{V}}$.

Then the commutator subgroup $[G, G]$ is uniformly perfect. Furthermore, if $\mathrm{fragd}_G^{\mathcal{V}} = \rho$ and $\mathrm{cld}_G = d$ then $\mathrm{cld}_{[G,G]} \leq d\rho^2$.

Uniform perfectness of [G,G]

Theorem 2

Let X be a paracompact topological space, let $G \leq \mathcal{H}(X)$ with cl_G bounded (as the norm on $[G, G]$) and let $\mathcal{U}$ be a G-invariant open cover of X such that

1. G is strongly $\mathcal{U}$-moving, and
2. there is an open cover $\mathcal{V}$ satisfying $\mathcal{V} \prec \mathcal{U}$ such that G is $\mathcal{V}$-factorizable and G is bounded with respect to the fragmentation norm $\mathrm{frag}^{\mathcal{V}}$.

Then the commutator subgroup $[G, G]$ is uniformly perfect. Furthermore, if $\mathrm{fragd}_G^{\mathcal{V}} = \rho$ and $\mathrm{cld}_G = d$ then $\mathrm{cld}_{[G,G]} \leq d\rho^2$.

Corollary 2

If $\mathcal{U}$ is a G-invariant open cover of X such that G is strongly $\mathcal{U}$-moving and $\mathcal{V}$-factorizable for some open cover $\mathcal{V}$ satisfying $\mathcal{V} \prec \mathcal{U}$ then $[G, G]$ is perfect.

Uniform perfectness of [G,G]

Theorem 3

Let X be a paracompact topological space, let $G \leq \mathcal{H}(X)$ with cl_G bounded, and let $\mathcal{U}$ be an open cover of X such that

1. G is $\mathcal{U}$-moving, and
2. there are G-invariant open covers $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{T}$ fulfilling the relation $\mathcal{T} \prec \mathcal{W} \prec \mathcal{V} \prec \mathcal{U}$, and such that G is $\mathcal{T}$-factorizable and it is bounded with respect to $\mathrm{frag}^{\mathcal{T}}$.

Then $[G, G]$ is uniformly perfect and $\mathrm{cld}_{[G,G]} \leq 4d\rho^3$ provided $\mathrm{fragd}_G^{\mathcal{T}} = \rho$ and $\mathrm{cld}_G = d$.

Uniform perfectness of [G,G]

Theorem 3

Let X be a paracompact topological space, let $G \leq \mathcal{H}(X)$ with cl_G bounded, and let $\mathcal{U}$ be an open cover of X such that

1. G is $\mathcal{U}$-moving, and
2. there are G-invariant open covers $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{T}$ fulfilling the relation $\mathcal{T} \prec \mathcal{W} \prec \mathcal{V} \prec \mathcal{U}$, and such that G is $\mathcal{T}$-factorizable and it is bounded with respect to $\mathrm{frag}^{\mathcal{T}}$.

Then $[G, G]$ is uniformly perfect and $\mathrm{cld}_{[G,G]} \leq 4d\rho^3$ provided $\mathrm{fragd}_G^{\mathcal{T}} = \rho$ and $\mathrm{cld}_G = d$.

Corollary 3

If G is $\mathcal{U}$-moving and $\mathcal{T}$-factorizable for some G-invariant open covers $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{T}$ such that $\mathcal{T} \prec \mathcal{W} \prec \mathcal{V} \prec \mathcal{U}$, then $[G, G]$ is perfect.

Theorem 4 (Epstein [2])

Let X be a paracompact space, let G be a group of homeomorphisms of X and let $\mathcal{B}$ be a basis of open sets of X satisfying the following axioms:
Axiom 1. If $U \in \mathcal{B}$ and $g \in G$, then $g(U) \in \mathcal{B}$.
Axiom 2. G acts transitively on $\mathcal{B}$ (i.e. $\forall\, U, V \in \mathcal{B}\, \exists\, g \in G : g(U) = V$).
Axiom 3. Let $g \in G$, $U \in \mathcal{B}$ and let $\mathcal{U} \subset \mathcal{B}$ be a cover of X. Then there exist an integer n, elements $g_1, \ldots, g_n \in G$ and $V_1, \ldots, V_n \in \mathcal{U}$ such that $g = g_n g_{n-1} \ldots g_1$, $\mathrm{supp}(g_i) \subset V_i$ and

$$\mathrm{supp}(g_i) \cup (g_{i-1} \ldots g_1(\overline{U})) \neq X \text{ for } 1 \leqslant i \leqslant n.$$

Then $[G, G]$, the commutator subgroup of G, is simple.

This famous theorem was published in 1970 and played a great role in investigation on diffeomorphism groups. By means of this theorem Thurston proved that $\mathcal{D}^{\infty}_c(M)$ is simple, Mather proved that $\mathcal{D}^{r}_c(M)$ is simple for $r \neq \dim M + 1$, Banyaga proved that the hamiltonian symplectomorphism group is simple. Recently Rybicki proved that identity component of contactomorphism group is simple.

We say that $G \leq \mathcal{H}(X)$ acts *transitively inclusively* (c.f. [3]) on a topological basis $\mathcal{B}$ if for all $U, V \in \mathcal{B}$ there is $g \in G$ such that $g(U) \subset V$.

Remark: The group of all diffeomorphisms which can be joined with identity by compactly supported isotopy acts transitively inclusively on the base of balls. On the other hand, volume preserving diffeomorphism group (and hence also the simplectomorphism group) does not.

Theorem 5 ([3])

Let X be a paracompact space, let $G \leq \mathcal{H}(X)$ and let $\mathcal{B}$ be a basis of open sets of X satisfying the following axioms:
Axiom 1. G acts transitively inclusively on $\mathcal{B}$.
Axiom 2. G is $\mathcal{U}$-factorizable for all covers $\mathcal{U} \subset \mathcal{B}$.
Then $[G, G]$ is a simple group.

Simplicity and uniform simplicty of $[G, G]$

A group G is called **uniformly simple** if there is $d > 0$ such that for all $f, g \in G$ with $f \neq e$ we have $g = h_1 \bar{f} h_1^{-1} \dots h_s \bar{f} h_s^{-1}$, where $s \leq d$, $\bar{f} = f$ or $\bar{f} = f^{-1}$, and $h_1, \dots, h_s \in G$. Given a uniformly simple group G, denote by usd_G the least d as above

Simplicity and uniform simplicty of $[G, G]$

A group G is called **uniformly simple** if there is $d > 0$ such that for all $f, g \in G$ with $f \neq e$ we have $g = h_1 \bar{f} h_1^{-1} \dots h_s \bar{f} h_s^{-1}$, where $s \leq d$, $\bar{f} = f$ or $\bar{f} = f^{-1}$, and $h_1, \dots, h_s \in G$. Given a uniformly simple group G, denote by usd_G the least d as above

Theorem 6

Let $\mathcal{B}$ be a topological basis of X. Suppose that $G \leq \mathcal{H}(X)$ satisfies the following conditions:

1. cl_G is bounded;
2. G acts transitively inclusively on $\mathcal{B}$;
3. there is an open cover $\mathcal{U} \prec \mathcal{B}$ such that G is $\mathcal{U}$-factorizable and G is bounded w.r.t. the fragmentation norm $\mathrm{frag}_G^{\mathcal{U}}$.

Then the group $[G, G]$ is uniformly simple. Moreover, if $\mathrm{cld}_G = d$ and $\mathrm{fragd}_G^{\mathcal{U}} = \rho$ then $\mathrm{usd}_G \leq 4d\rho^2$.

Simplicity and uniform simplicty of $[G, G]$

A group G is called **uniformly simple** if there is $d > 0$ such that for all $f, g \in G$ with $f \neq e$ we have $g = h_1 \bar{f} h_1^{-1} \dots h_s \bar{f} h_s^{-1}$, where $s \leq d$, $\bar{f} = f$ or $\bar{f} = f^{-1}$, and $h_1, \dots, h_s \in G$. Given a uniformly simple group G, denote by usd_G the least d as above

Theorem 6

Let $\mathcal{B}$ be a topological basis of X. Suppose that $G \leq \mathcal{H}(X)$ satisfies the following conditions:

1. cl_G is bounded;
2. G acts transitively inclusively on $\mathcal{B}$;
3. there is an open cover $\mathcal{U} \prec \mathcal{B}$ such that G is $\mathcal{U}$-factorizable and G is bounded w.r.t. the fragmentation norm $\mathrm{frag}_G^{\mathcal{U}}$.

Then the group $[G, G]$ is uniformly simple. Moreover, if $\mathrm{cld}_G = d$ and $\mathrm{fragd}_G^{\mathcal{U}} = \rho$ then $\mathrm{usd}_G \leq 4d\rho^2$.

Corollary 6

If $G \leq \mathcal{H}(X)$ is factorizable and bounded, and G acts transitively inclusively on some basis $\mathcal{B}$ of X, then $[G, G]$ is uniformly simple.

1. Let M be a manifold with a boundary, $\dim(M) = n \geqslant 2$. Then $G = \mathcal{D}^r_c(M)$, where $r = 0, 1, \ldots, \infty$, $r \neq n$ and $r \neq n+1$ is perfect and non-simple.
Likewise, let N be a submanifold of M of class C^r, $r = 0, 1, \ldots, \infty$, and $\dim N \geq 1$. It was proved by Rybicki that G_c, where $G = \mathcal{D}^r(M, N)$ is the identity component of the group of C^r-diffeomorphisms preserving N, is perfect. G_c is clearly non-simple.

Groups $\mathcal{D}^r_c(H^+)$ (where $H^+ = \{(x_1, \ldots, x_n) : x_n \geqslant 0\}$ and $r \neq n$, $r \neq n+1$) and $\mathcal{D}^r_c(\mathbb{R}^n, \mathbb{R}^{n-1} \times \{0\})$ fulfilled assumptions of Theorem 1, so they are uniformly perfect .
For $r = n$ or $r = n+1$ groups $[\mathcal{D}^r_c(H^+), \mathcal{D}^r_c(H^+)]$ and $[\mathcal{D}^r_c(\mathbb{R}^n, \mathbb{R}^{n-1} \times \{0\}), \mathcal{D}^r_c(\mathbb{R}^n, \mathbb{R}^{n-1} \times \{0\})]$ are uniformly perfect.

Examples

2. Given a foliation $\mathcal{F}$ of dimension k on a manifold M, let $G = \mathcal{D}^r(M, \mathcal{F})$ be the identity component of the group of all diffeomorphisms of class C^r taking each leaf to itself. Due to results of Rybicki, Fukui and Imanishi and Tsuboi, the group G_c is perfect provided $r = 0, 1, \dots, k$ or $r = \infty$. It is very likely that for large (but finite) r the group $\mathcal{D}^r_c(M, \mathcal{F})$ is not perfect. By Ling's theorem $[\mathcal{D}^r_c(M, \mathcal{F}), \mathcal{D}^r_c(M, \mathcal{F})]$ is perfect for all r. It is a highly non-trivial problem whether G_c is uniformly perfect. Several results of our studies apply to G_c.

2. Given a foliation $\mathcal{F}$ of dimension k on a manifold M, let $G = \mathcal{D}^r(M, \mathcal{F})$ be the identity component of the group of all diffeomorphisms of class C^r taking each leaf to itself. Due to results of Rybicki, Fukui and Imanishi and Tsuboi, the group G_c is perfect provided $r = 0, 1, \dots, k$ or $r = \infty$. It is very likely that for large (but finite) r the group $\mathcal{D}^r_c(M, \mathcal{F})$ is not perfect. By Ling's theorem $[\mathcal{D}^r_c(M, \mathcal{F}), \mathcal{D}^r_c(M, \mathcal{F})]$ is perfect for all r. It is a highly non-trivial problem whether G_c is uniformly perfect. Several results of our studies apply to G_c.

3. Assume now that $\mathcal{F}$ is a singular foliation, i.e. the dimensions of its leaves need not be equal. One can consider the group of leaf-preserving diffeomorphisms of $\mathcal{F}$, $G = \mathcal{D}^\infty_c(M, \mathcal{F})$. However, it is hopeless to obtain any perfectness results for this group. On the other hand, Ling's Theorem still works in this case and we know that the commutator group $[G_c, G_c]$ is perfect. We do not know whether $[G_c, G_c]$ is uniformly perfect.

2. Given a foliation $\mathcal{F}$ of dimension k on a manifold M, let $G = \mathcal{D}^r(M, \mathcal{F})$ be the identity component of the group of all diffeomorphisms of class C^r taking each leaf to itself. Due to results of Rybicki, Fukui and Imanishi and Tsuboi, the group G_c is perfect provided $r = 0, 1, \ldots, k$ or $r = \infty$. It is very likely that for large (but finite) r the group $\mathcal{D}^r_c(M, \mathcal{F})$ is not perfect. By Ling's theorem $[\mathcal{D}^r_c(M, \mathcal{F}), \mathcal{D}^r_c(M, \mathcal{F})]$ is perfect for all r. It is a highly non-trivial problem whether G_c is uniformly perfect. Several results of our studies apply to G_c.

3. Assume now that $\mathcal{F}$ is a singular foliation, i.e. the dimensions of its leaves need not be equal. One can consider the group of leaf-preserving diffeomorphisms of $\mathcal{F}$, $G = \mathcal{D}^\infty_c(M, \mathcal{F})$. However, it is hopeless to obtain any perfectness results for this group. On the other hand, Ling's Theorem still works in this case and we know that the commutator group $[G_c, G_c]$ is perfect. We do not know whether $[G_c, G_c]$ is uniformly perfect.
Any Jacobi manifold (M, Λ, E) (and in particular, for $E = 0$, Poisson manifold) produces a singular foliation $\mathcal{F}(\Lambda, E)$. So that similar problems arise for some automorphism groups of a Jacobi manifold which are subgroups of $\mathcal{D}(M, \mathcal{F}(\Lambda, E))$.

References

[1] D.Burago, S.Ivanov and L.Polterovich, *Conjugation invariant norms on groups of geometric origin*, Advanced Studies in Pures Math. 52, Groups of Diffeomorphisms (2008), 221-250.

[2] D.B.A.Epstein, *The simplicity of certain groups of homeomorphisms*, Compositio Mathematica 22, Fasc.2 (1970), 165-173.

[3] W.Ling, *Factorizable groups of homeomorphisms*, Compositio Mathematica, 51 no. 1 (1984), p. 41-50.

[4] I.Michalik, T.Rybicki, *On the structure of the commutator subgroup of certain homeomorphism groups*, Topology and its Applications, 158 (2011) 1314Ű1324.