THE CANONICAL EIGHT-FORM ON MANIFOLDS WITH RESTRICTED HOLONOMY GROUP SPIN(9)

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Berger's list of restricted holonomy groups

Holonomy groups of Riemannian non locally symmetric manifolds:

- $\mathbf{SO(n)}$ – parallel differential forms (Par.Dif.Forms) do not exist;
- $\mathbf{U(m)}$, $n = 2m, m \geqslant 2$ – Par.Dif.Forms – powers of the Kähler form (up to a non-zero factor);
- $\mathbf{SU(m)}$, $n = 2m, m \geqslant 2$ – Par.Dif.Forms – powers of the Kähler form, complex volume form and its conjugated;
- $\mathbf{Sp(1)Sp(m)}$, $n = 4m, m \geqslant 2$ – Par.Dif.Forms – powers of the invariant 4-form;
- $\mathbf{Sp(m)}$, $n = 4m, m \geqslant 2$ – Par.Dif.Forms – generated by tree Kähler forms;
- $\mathbf{Spin(9)}$, $n = 16$ – Par.Dif.Forms - exists only one parallel 8-form;
- $\mathbf{Spin(7)}$, $n = 8$ – Par.Dif.Forms – exists only one parallel 4-form (E. Bonan);
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The group $\mathrm{Spin}(9)$ belongs to Berger's list of restricted holonomy groups of locally irreducible Riemannian manifolds which are not locally symmetric. Manifolds with holonomy group $\mathrm{Spin}(9)$ have been studied by Alekseevsky (1968), Brown and Gray (1972), Friedrich (2001,2003), and Lam (2008), among other authors. As proved by Alekseevsky

D. V. Alekseevskiĭ, Riemannian spaces with non-standard holonomy groups, *Funct. Anal. Appl.* **2** (1968), 97–105.

and by Brown and Gray

R. B. Brown and A. Gray, Riemannian manifolds with holonomy group $\mathrm{Spin}(9)$, *Diff. Geom. in honor of K. Yano*, (Kinokuniya, Tokyo, 1972), pp. 41–59.

a connected, simply-connected, complete non-flat $\mathrm{Spin}(9)$-manifold is isometric to either the Cayley projective plane

$$\mathbb{O}\mathrm{P}(\mathbf{2}) \cong \mathbf{F_4}/\mathrm{Spin}(\mathbf{9})$$

or its dual symmetric space, the Cayley hyperbolic plane

$$\mathbb{O}\mathrm{H}(\mathbf{2}) \cong \mathbf{F_{4(-20)}}/\mathrm{Spin}(\mathbf{9}).$$

Let $\mathbf{\Delta_9}$ be the unique irreducible 16-dimensional $\mathrm{Spin}(9)$-module.

The $\mathrm{Spin}(9)$-module $\Lambda^8(\Delta_9^*)$

contains one and only one (up to a non-zero factor)

8-form Ω_0^8

which is $\mathrm{Spin}(9)$-invariant and defines the unique parallel form on $\mathbb{O}\mathrm{P}(2)$. It induces

a *canonical* 8*-form* Ω^8

on any 16-dimensional manifold with a fixed $\mathrm{Spin}(9)$-structure. This form is said to be canonical because (R. B. Brown and A. Gray (1972), Berger (1972)) it yields, for the compact case, a generator of

$$H^8(\mathbb{O}\mathrm{P}(2), \mathbb{R}).$$

Some explicit expressions of Ω_0^8 have been given.

- The first one by Brown and Gray (1972) in in terms of a Haar integral over $Spin(8)$:

$$\Omega_0^8 = \int_{Spin(8)} (*),$$

where expr. $(*)$ – vol. form in $\mathbb{O} = \mathbb{R}^8$ and act. of $Spin(8)$ in $\mathbb{R}^{16} = \mathbb{O}^2$;

- in the same year 1972 when the quoted paper by R. Brown and A. Gray appeared, Berger published an article on the Riemannian geometry of rank one symmetric spaces

M. Berger, *Du côté de chez Pu*, Ann. Sci Ecole Norm. Sup. 5 (1972), 1-44.

containing the following integral definition of a $Spin(9)$-invariant 8-form Ω_0^8 in $\mathbb{R}^{16}$:

$$\Omega_0^8 = c \int_{\mathbb{S}^8} (**);$$

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A natural question is whether an explicit and possibly simple algebraic expression for Ω_0^8 can be written in $\mathbb{O}^2 = \mathbb{R}^{16}$, in parallel with the usual definitions of the G_2-invariant 3-form on $\mathbb{R}^7$ or the $Spin(7)$-invariant 4-form on $\mathbb{R}^8$:
see the article of E. Bonan

E. Bonan, Sur les variétés riemanniennes à groupe d'holonomie G_2 ou $\mathrm{Spin}(7)$, *C. R. Acad. Sci. Paris Sér. I Math.* **262** (1966), 127–129.

or for example the books

D. D. Joyce, Compact Manifolds with Special Holonomy, Oxford University Press, 2000.

D. D. Joyce, Riemannian Holonomy Groups and Calibrated Geometry, Oxford University Press, 2007.

Indeed, some such algebraic expressions have already been written.

- The expression for the form Ω_0^8 was then given by Brada and Pécaut-Tison in the following papers

C. Brada and F. Pécaut-Tison, Calcul explicite de la courbure et de la 8-forme canonique du plan projectif des octaves de Cayley, *C. R. Acad. Sci. Paris Sér. I Math.* **301** (1985), no. 2, 41–44.

and

C. Brada and F. Pécaut-Tison, Géométrie du plan projectif des octaves de Cayley, *Geom. Dedicata* **23** (1987), no. 2, 131–154.

by using a "cross product." Unfortunately, their formula is not correct, as we explain in the Appendix I.

- Another expression was then given by Abe and Matsubara in two papers

Announce

K. Abe, Closed regular curves and the fundamental form on the projective spaces, *Proc. Japan Acad. Ser. A Math. Sci.* **68** (1992), no. 6, 123–125.

K. Abe and M. Matsubara, Invariant forms of the exceptional symmetric spaces FII and $EIII$, *Transformation group theory*, (Taejŏn, 1996), 3–16, Korea Adv. Inst. Sci. Tech., Taejŏn.

as a sum of 702 suitable terms. Their formula contains some errors: Abe and Matsubara attempted to describe this 702-terms expression for Ω_0^8. The form Ω_0^8 is exhibited there as a sum of eight 8-forms $\Omega_1^8, \ldots, \Omega_8^8$. But the combinatorial definitions of these eight 8-forms contain some mistakes, for example the definition of the form Ω_8^8 is not correct. Moreover, the first papers and the second paper contain different expressions for the aforementioned form Ω_8^8 (the definitions of $\Omega_1^8, \ldots, \Omega_7^8$ coincide). The expression given in 1-st p. contains at most $7 \cdot 7 \cdot 4 = 196$ terms (in some canonical basis) though it is asserted in the 2-d p. that Ω_8^8 contains 336 terms.

We give an explicit expression of the canonical 8-form Ω^8 on a $\mathrm{Spin}(9)$-manifold, in terms of the nine local symmetric involutions involved.

A major progress in understanding $Spin(9)$-structures came in the context of weak holonomies by the work of Th. Friedrich: in papers

Th. Friedrich, Weak $\mathrm{Spin}(9)$-structures on 16-dimensional Riemannian manifolds, *Asian J. Math.* **5** (2001), no. 1, 129–160.

Th. Friedrich, $\mathrm{Spin}(9)$-structures and connections with totally skew-symmetric torsion, *J. Geom. Phys.* **47** (2003), no. 2–3, 197–206.

it is observed that the number of possible "weakened" holonomies $Spin(9)$ is 16, exactly like in the cases of the groups $U(n)$ and G_2, and also that a Spin(9)-structure on M^{16} can be described as a certain vector subbundle ν^9 of the bundle of endomorphisms $\mathrm{End}(TM)$. This fact suggests a similarity between $Spin(9)$ and the quaternionic group $Sp(n)Sp(1)$.

More precisely, a $\mathrm{Spin}(9)$-structure on an connected, oriented 16-dimensional Riemannian manifold (M, g) is defined by a nine-dimensional subbundle ν^9 of the bundle of endomorphisms $\mathrm{End}(TM)$ locally spanned by $I_i \in \Gamma(\nu^9)$, $0 \leqslant i \leqslant 8$, satisfying the relations

$$\mathbf{I_i I_j + I_j I_i = 0}, \quad \mathbf{i \neq j}, \qquad \mathbf{I_i^2} = \mathrm{Id}, \qquad \mathbf{I_i^T = I_i}, \qquad \mathrm{tr}\, \mathbf{I_i = 0},$$

$i, j = 0, \ldots, 8$. These endomorphisms define 2-forms ω_{ij}, $0 \leqslant i < j \leqslant 8$, on M locally by

$$\omega_{\mathbf{ij}}(\mathbf{X}, \mathbf{Y}) = \mathbf{g}(\mathbf{X}, \mathbf{I_i I_j Y}).$$

Similarly, using the skew-symmetric involutions $I_i I_j I_k$, $0 \leqslant i < j < k \leqslant 8$, one can define 2-forms σ_{ijk}. The 2-forms $\{\omega_{\mathbf{ij}}, \sigma_{\mathbf{ijk}}\}$ are linearly independent and a local basis of the bundle $\Lambda^2 M$. Remark that the local tensors $I_{ij} = I_i I_j$ and $I_{ijk} = I_i I_j I_k$ define local almost complex structures on M^{16}.

- Our main result

Theorem (Main)

The canonical 8*-form on the* $\mathrm{Spin}(9)$*-manifold* (M, g, ν^9) *is given by*

$$\Omega^8 = \sum_{\substack{i,j=0,\dots,8 \\ i',j'=0,\dots,8}} \omega_{ij} \wedge \omega_{ij'} \wedge \omega_{i'j} \wedge \omega_{i'j'},$$

where $\omega_{ij} = -\omega_{ji}$ *if* $i > j$ *and* $\omega_{ij} = 0$ *if* $i = j$.
The stabilizer group of the canonical 8*-form* Ω^8_0 *on* $\mathbb{R}^{16}$*, under the natural action of the group* $\mathrm{GL}(16, \mathbb{R})$*, is the Lie group* $\rho(\mathrm{Spin}(9)) \subset \mathrm{GL}(16, \mathbb{R})$.

Note that the sequence $ij, ij', i'j, i'j' \in \overline{D}$ is a sequence of vertices of either a rectangle or a degenerate rectangle (a point, an interval) made of entries of a square 9×9 matrix.

Castrillón López M., Gadea P. M., Mykytyuk I.V. *The canonical eight-form on manifolds with holonomy group Spin(9).* International Journal of Geometric Methods in Modern Physics, **7**, 7,(2010), pp. 1159–1183.

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M. Parton and P. Piccinni Spin(9) AND ALMOST COMPLEX STRUCTURES ON 16-DIMENSIONAL MANIFOLDS

where the following expression for the invariant form was obtained

$$\Omega^8 = c \cdot \sum_{0 \le i_1 < i_2 < i_3 < i_4 \le 8} (\omega_{i_1 i_2} \wedge \omega_{i_3 i_4} - \omega_{i_1 i_3} \wedge \omega_{i_2 i_4} + \omega_{i_1 i_4} \wedge \omega_{i_2 i_3})^2.$$

Moreover,

$$\Omega^8 = c \cdot \tau_4(\omega)$$

where $\tau_4(\omega)$ is the fourth coefficient of the characteristic polynomial of the skew-symmetric matrix of Kähler 2-forms
$\omega = (\omega_{ij}), i, j = 0, .., 8.$
The computation was performed with the help of the software "Mathematica", in particular, all 702 terms of the $Spin(9)$-invariant form are described in this article. The calculations are based on the Berger integral formula for the form Ω^8.

Corollaries to Main theorem

We can get some consequences of the proof of Main theorem. The 4-form $\sum_{i,j=0,\dots,8} \varpi_{ij} \wedge \varpi_{ij}$ on the space $T_pM \equiv \mathbb{O}^2$ is invariant with respect to the action of the Lie group $\mathrm{Spin}(9)$. It is $\mathrm{Spin}(9)$-invariant hence trivial, so it defines a global (trivial) 4-form on M. We thus obtain the next corollary to Main theorem.

Corollary

The 4-form $\sum_{0\leqslant i<j\leqslant 8} \omega_{ij} \wedge \omega_{ij} = 0$, vanishes, i.e. we have

$$\sum_{0\leqslant i<j\leqslant 8} \{\omega_{ij}(X,Y)\omega_{ij}(Z,W) - \omega_{ij}(X,Z)\omega_{ij}(Y,W) + \omega_{ij}(Y,Z)\omega_{ij}(X,W)\} = 0,$$

or, equivalently,

$$\mathfrak{S}_{XYZ} \sum_{0\leqslant i<j\leqslant 8} \omega_{ij}(X,Y)W^\flat(I_{ij}Z) = 0, \qquad X,Y,Z,W \in \mathfrak{X}(M). \quad (1)$$

Moreover, since the 8-form $\left(\sum_{i,j=0,\dots,8} \varpi_{ij} \wedge \varpi_{ij}\right) \wedge \left(\sum_{i',j'=0,\dots,8} \varpi_{i'j'} \wedge \varpi_{i'j'}\right)$ vanishes, we can rewrite the expression of the canonical form as

Corollary

$$\Omega^8 = -\frac{1}{2} \sum_{\substack{i,j=0,\dots,8 \\ i',j'=0,\dots,8}} \left(\omega_{ij} \wedge \omega_{i'j'} - \omega_{i'j} \wedge \omega_{ij'}\right)^2.$$

Consider now the forms σ_{ijk}, $i, j, k = 0, \dots, 8$, being zero if at least a couple of indices are equal, and satisfying $\sigma_{\pi(i),\pi(j),\pi(k)} = \operatorname{sgn}(\pi)\sigma_{ijk}$ when $i \neq j \neq k \neq i$, for any permutation $\pi \in S_3$.
The 4-form $\sum_{i,j,q=0,\dots,8} \bar{\sigma}_{ijq} \wedge \bar{\sigma}_{ijq}$ on the space $T_pM \equiv \mathbb{O}^2$ is invariant with respect to the action of the Lie group $\mathrm{Spin}(9)$. It is $\mathrm{Spin}(9)$-invariant and, consequently, it is also trivial ([**?**, Sect. 5]), so we obtain

Corollary

The 4-form $\sum_{0\leqslant i<j<k\leqslant 8} \sigma_{ijk} \wedge \sigma_{ijk}$, vanishes, i.e. we have

$$\sum_{0\leqslant i<j<k\leqslant 8} \{\sigma_{ijk}(X,Y)\sigma_{ijk}(Z,W) - \sigma_{ijk}(X,Z)\sigma_{ijk}(Y,W) + \sigma_{ijk}(Y,Z)\sigma_{ijk}(X,W)\} = 0.$$

Using the method of the proof of Main theorem one could obtain the expression for the canonical form Ω^8 in terms of the 2-forms σ_{ijp}. But since the proof is technically more complicated, we state it as the next

Conjecture. *The canonical* 8*-form* Ω^8 *on the* $\mathrm{Spin}(9)$*-manifold* (M, g, ν^9) *is given by*

$$\Omega^8 = \frac{1}{4} \sum_{\substack{i,j=0,\dots,8 \\ i',j'=0,\dots,8}} \sum_{p,p'=0,\dots,8} \sigma_{ijp} \wedge \sigma_{ijp'} \wedge \sigma_{i'j'p} \wedge \sigma_{i'j'p'}.$$

Note. The Conjecture has been recently solved in the affirmative by E. Bonan by using his own, different methods (personal communication).

The Curvature Tensor of the Cayley Planes

We now apply our previous conclusions to obtain an expression of the curvature tensor of the Cayley planes in terms of the nine local symmetric involutions involved and then to relate it to the well-known expression in terms of triality given by Brown and Gray, to the one in terms of the brackets of the Lie algebra $\mathfrak{f}_4$ of F_4, furnished by Brada and Pécaut-Tison, and also to the expression given in

I. V. Mykytyuk, The triple Lie system of the symmetric space $F_4/\mathrm{Spin}(9)$, *Asian J. Math.* **6** (2002), no. 4, 713–718.

First recall (Alekseevskij, Brown and Gray) that the curvature tensor R of a non-flat $\mathrm{Spin}(9)$-manifold is a non-zero multiple of the curvature tensor $R^{\mathbb{O}\mathrm{P}(2)}$ of $\mathbb{O}\mathrm{P}(2)$. Further, as duality reverses curvature, in the next formulas we can take a constant $c \in \mathbb{R}\backslash\{0\}$, being understood that $c > 0$ (resp. $c < 0$) in the compact (resp. noncompact) case.
Then we have

Theorem

The curvature tensor $R_{XY}Z$ of the Cayley planes is given by

$$R_{XY}Z = -\frac{c}{4} \sum_{0 \leqslant i<j \leqslant 8} \omega_{ij}(X,Y) I_{ij} Z, \qquad c \in \mathbb{R}\backslash\{0\}. \tag{2}$$

Remark only that the form $\lambda \sum_{i<j} \omega_{ij} \otimes I_{ij}$, $\lambda \in \mathbb{R}$, is a $\rho_*(\mathfrak{spin}(9))$-valued 2-form. Moreover, the necessary algebraic conditions are clearly satisfied by $\lambda \sum_{i<j} \omega_{ij} \otimes \omega_{ij}$, except for the Bianchi identity, but this is immediate from Corollary. Taking $\lambda = -\frac{c}{4}$, we see that the absolute value of the sectional curvature belongs to $[|c|/4, |c|]$.

Brown and Gray give in [**?**, (6.12)] an explicit expression for the curvature tensor $R_{XY}Z$ of $\mathbb{O}\mathrm{P}(2)$.
Letting $\mathbb{R}^{16} \equiv \mathbb{O}^2$, Brown and Gray's formula for the curvature tensor can be written as $R_{XY}Z = S_{XY}Z - S_{YX}Z$, where

$$S_{XY}Z = -\frac{c}{4}\big(4\langle y_1, z_1\rangle x_1 + (z_1y_2)\bar{x}_2 + (x_1y_2)\bar{z}_2, \\ 4\langle y_2, z_2\rangle x_2 + \bar{x}_1(y_1z_2) + \bar{z}_1(y_1x_2)\big),$$

for $X = (x_1, x_2)$, $Y = (y_1, y_2)$, $Z = (z_1, z_2) \in \mathbb{O}^2$.
However, in our paper a simple expression for either $R^{\mathbb{O}\mathrm{P}(2)}$ or $R^{\mathbb{O}\mathrm{H}(2)}$ has been given in terms of the nine local symmetric operators. We can write it as $R_{XY}Z = S'_{XY}Z - S'_{YX}Z$, where

$$S'_{XY}Z = -\frac{c}{4}\Big(3g(Y,Z)X + \sum_{0\leqslant i\leqslant 8} g(I_iY, Z)I_iX\Big).$$

This expression, in terms of the octonion algebra has the following form (Mykytyuk, 2002) for $X = (x_1, x_2)$, $Y = (y_1, y_2)$ and $Z = (z_1, z_2)$,

$$S'_{XY}Z = -\frac{c}{4}\big((x_1\bar{y}_1)z_1 + (x_1y_2)\bar{z}_2 + (z_1\bar{y}_1)x_1 + (z_1y_2)\bar{x}_2, \\ \bar{z}_1(y_1x_2) + z_2(\bar{y}_2x_2) + x_2(\bar{y}_2z_2) + \bar{x}_1(y_1z_2)\big).$$

Then we have

Theorem

The curvature tensor of the Cayley planes is given by either the expression of the theorem above or any of those obtained using $S'_{XY}Z$ or $S_{XY}Z$. Moreover, one has

$$R_{XY}Z = \frac{1}{5}\sum_{0\leqslant j\leqslant 8} I_jR_{XY}I_jZ.$$

The action of $\mathrm{Spin}(9)$ on $\mathbb{R}^{16} \equiv \mathbb{O}^2$

The isotropy representation of either $\mathbb{O}\mathrm{P}(2)$ or $\mathbb{O}\mathrm{H}(2)$ is known to be isomorphic to the 16-dimensional spin representation ρ of $\mathrm{Spin}(9)$.
Let V^9 be a real vector space of dimension nine endowed with a positive definite bilinear form Q. Let $e_0, \ldots, e_8$ be an orthonormal basis of V^9.
The Clifford algebra $\mathrm{Cl}_+(9)$ in terms of this basis is defined as the real associative algebra with unit 1, generators $e_0, \ldots, e_8$, and defining relations

$$e_i \cdot e_j + e_j \cdot e_i = 0, \quad i \neq j, \qquad e_i^2 = 1, \quad i, j = 0, \ldots, 8.$$

Let $\mathrm{Pin}_+(9)$ be the multiplicative subgroup of the group of all the invertible elements of $\mathrm{Cl}_+(9)$ generated by the vectors of length one in V^9. If $Q(v,v) = 1$ then $v \cdot v = 1$, so $v \in \mathrm{Pin}_+(9)$. The Lie group $\mathrm{Spin}_+(9)$, which we denote simply by $\mathrm{Spin}(9)$, as they are isomorphic, is the subgroup of $\mathrm{Pin}_+(9)$ consisting of even elements, i.e.

$$\mathrm{Spin}(9) = \{\, v_1 \cdot v_2 \cdot \ldots \cdot v_{2k},\ Q(v_i, v_i) = 1,\ i = 1, \ldots, 2k,\ k \in \mathbb{N}\,\}.$$

Moreover, the group $\mathrm{Spin}(9)$ preserves under conjugation the space V^9, that is, $sV^9s^{-1} = V^9$ for all $s \in \mathrm{Spin}(9)$. We denote by π the corresponding representation of the group $\mathrm{Spin}(9)$ on V^9. Then $\pi(\mathrm{Spin}(9)) = \mathrm{SO}(9)$ and $\pi\colon \mathrm{Spin}(9) \to \mathrm{SO}(9)$ is the usual two-fold covering homomorphism.

There exists a faithful representation ρ of $\mathrm{Pin}_+(9)$ by orthogonal matrices. In other words, $\rho(\mathrm{Pin}_+(9)) \subset \mathrm{O}(16)$ and $\rho(\mathrm{Spin}(9)) \subset \mathrm{SO}(16)$. Therefore, there exist nine orthogonal linear transformations I_i of $\Delta_9 = \mathbb{R}^{16}$ satisfying the relations

$$I_iI_j+I_jI_i = 0,\ i \neq j, \quad I_i^2 = \mathrm{I}, \quad I_i^T = I_i, \quad \mathrm{tr}\ I_i = 0, \qquad i,j = 0,\dots,8.$$

The set $\{I_iI_j, 0 \leqslant i < j \leqslant 8\}$ is a basis of the Lie algebra $\rho_*(\mathfrak{spin}(9)) \subset \mathfrak{so}(16)$ because

$$[I_iI_j, I_k] = \begin{cases} 0, & \text{if } k \neq i,j, \\ -2I_j, & \text{if } k = i, \\ 2I_i, & \text{if } k = j, \end{cases}$$

The operators I_iI_j are linearly independent and generate a space of dimension equal to $\dim \mathfrak{so}(9)$.

Taking into account that each operator $I_i I_j$ is the tangent vector at $t = 0$ to the curve

$$s(t) = \big(\cos(t/2)I_i - \sin(t/2)I_j\big)\big(\cos(t/2)I_i + \sin(t/2)I_j\big) = \cos t \cdot \mathrm{I} + \sin t \cdot I_i I_j$$

in $\rho(\mathrm{Spin}(9))$ passing through the identity I, we obtain that the operators $I_i I_j$ generate the Lie algebra $\rho_*(\mathfrak{spin}(9))$ and, consequently, by the connectedness of the Lie group $\mathrm{Spin}(9)$ the following proposition holds

The Lie group $\rho(\mathrm{Spin}(9)) \subset \mathrm{SO}(16)$ is generated by the one-parameter families of endomorphisms

$$\exp(tI_iI_j) = \cos t \cdot \mathrm{I} + \sin t \cdot I_i I_j, \qquad 0 \leqslant i < j \leqslant 8, \quad t \in \mathbb{R}.$$

On this lemma is based the proof of the invariance of the form Ω_0^8:

$$M_{kl}^t \cdot \omega = A \cdot \omega \cdot A^{-1}, \quad k < l,$$

where A is orthogonal matrix with elements $a_{kk} = \cos(2t)$, $a_{lk} = \sin(2t)$, $a_{ll} = \cos(2t)$, $a_{kl} = -\sin(2t)$, $a_{ii} = 1 (i \neq k, l)$, $a_{ij} = 0 (i \neq j)$.

Nontriviality

Using the automorphism group of $\mathbb{O}$ (the Lie group G_2) and the properties of the form of the type $\omega_{ij} \wedge \omega_{ij'} \wedge \omega_{i'j} \wedge \omega_{i'j'}$ we reduce the problem to calculations of some coefficient of only three 8-forms with $(i, i'; j, j')$ equal to $(0, 0; 1, 2)$ or $(0, 1; 2, 3)$ or $(0, 1; 2, 4)$.

Appendix

Note also that the proof of the invariance of the form ω in the papers of C. Brada and F. Pécaut-Tison contains some gaps.

- First of all this proof is based on the wrong Proposition 5. The proof of this proposition relies in turn on the fact that the orthogonal transformations $T_a \colon \mathbb{O} \to \mathbb{O}$, $x \mapsto axa$, of the space $\mathbb{O}$, where $a \in \mathrm{Im}\,\mathbb{O}$, $a^2 = -1$, are pure imaginary octonions of length 1, generate a group G_T isomorphic to $\mathrm{SO}(8)$. But this is impossible because $T_a(u_0) = -u_0$ so that for any $g \in G_T$ we have $g(u_0) = \pm u_0$. Thus G_T is locally isomorphic to $\mathrm{SO}(7)$ so that $G_T \ncong \mathrm{SO}(8)$.

- Moreover, this Prop.5 asserts that the group G^* generated by certain one-parameter subgroup and by the orthogonal transformations $\tilde{T}_a\colon \mathbb{O}^2 \to \mathbb{O}^2$, $(x_1, x_2) \mapsto (ax_1, x_2 a)$, where $a \in \operatorname{Im}\mathbb{O}$, $a^2 = -1$, are purely imaginary octonions of length 1, is isomorphic to the group $\mathrm{Spin}(9)$. Now remark that their 8-form $\omega = \omega' \wedge \omega'$, i.e. it is the square of some 4-form ω', and it is proved that this 4-form ω' is G^*-invariant. But we know (Brown and Gray) that such a non-zero $\mathrm{Spin}(9)$-invariant 4-form cannot exist, so that $G^* \ncong \mathrm{Spin}(9)$.
- The form ω is not $Spin(9)$-invariant becouse for the operator I_{78} (which is an element of the Lie algebra $\rho_*(\mathfrak{spin}(9)) \subset \mathfrak{so}(16)$) and vectors $U_1, \dots U_8 \in \mathbb{O}^2$, $U_1 = (0, u_0)$ and $U_2 = (u_0, 0), \dots, U_8 = (u_6, 0)$, the following expression

$$\omega(I_{78}U_1, U_2, \dots, U_8) + \omega(U_1, I_{78}U_2, \dots, U_8) + \cdots + \omega(U_1, U_2, \dots, I_{78}U_8)$$

does not vanish.