Double structures and jets

D. J. Saunders

In honour of W. M. Tulczyjew

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Abstract

Double vector bundles, of which the double tangent bundle is a standard example, were introduced by Pradines in 1968.

In this talk I shall discuss how certain double vector bundles and other double structures can be used in the construction of jet bundles.

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The fibrations found in several jet structures are related in much the same way as V, A_α and PV

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submanifold of regular m-velocities

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The prolongation of $f : E_1 \to E_2$,

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$$\mathring{T}^{\mathrm{T}}_m f = \{ j_0\gamma \in \mathring{T}_m E_1 : f \circ \gamma \text{ an immersion} \}$$
$$T_m f : \mathring{T}^{\mathrm{T}}_m f \to \mathring{T}_m E_2 \,.$$

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Define $\pi_{mE} : J_m E \to E$ by $\pi_{mE}([j_0\gamma]) = \gamma(0) = \tau_{mE}(j_0\gamma)$.

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Also define the manifold of jets of local sections

$$J\pi = \{ j_x \sigma : x \in W \subset M \,, \quad \sigma : W \to E \,, \quad \pi \circ \sigma = \mathrm{id} \}$$

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$$j_x\sigma \mapsto [j_0(\sigma \circ \psi^{-1})]$$

where ψ is any chart defined around $x \in M$ with $\psi(x) = 0 \in \mathbb{R}^m$.

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where ψ is any chart defined around $x \in M$ with $\psi(x) = 0 \in \mathbb{R}^m$. This is a well-defined diffeomorphism.

Double velocities

The double velocity manifold is

$$T_m T_m E = \{ j_0 \tilde{\gamma} : \tilde{\gamma} \text{ an } m\text{-curve in } T_m E \}$$

and is a double vector bundle

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Vertical double velocity manifold

$$V_m \tau_{mE} = \{ j_0 \tilde{\gamma} : T_m \tau_{mE}(j_0 \tilde{\gamma}) = 0 \}$$

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Prolongations and semiprolongations

If γ is an m-curve in E then $\tilde{j}\gamma$,

$$\tilde{j}\gamma(t) = j_t\gamma = j_0(\gamma \circ \mathrm{t}_t)$$

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An m-curve $\tilde{\gamma}$ in $T_m E$ is a semiprolongation if

$$\tilde{\gamma}(0) = \tilde{j}(\tau_{mE} \circ \tilde{\gamma})(0) = j_0(\tau_{mE} \circ \tilde{\gamma}) .$$

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The submanifold $\hat{T}^2_m E = \{j_0\tilde{\gamma} \in T_m T_m E, \text{semiholonomic}\}$ defines an affine sub-bundle of the vector bundle $\tau_{m(T_m E)} : T_m T_m E \to E$, modelled on the vector bundle

$$V_m \tau_{mE} \cong \tau^*_{mE} TE \otimes_{T_m E} \bigotimes^2 \mathbb{R}^{m*} \to T_m E\,.$$

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The exchange map $e : T_m T_m E \to T_m T_m E$ is well-defined by $\psi \mapsto \hat{\psi}$ where $\hat{\psi}(t,s) = \psi(s,t)$.

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The map $\{j^2_0 \gamma\} \to T^2_m E$, $j^2_0\gamma \mapsto j_0(\tilde{j}\gamma)$ is a bijection.

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Vertical double velocities have symmetric and skew-symmetric components: $V_m \tau_{mE} \cong V^{\vee}_m \tau_{mE} \oplus_{T_m E} V^{\wedge}_m \tau_{mE}$,

$$V^{\vee}_m \tau_{mE} \cong \tau^*_{mE} TE \otimes_{T_m E} S^2 \mathbb{R}^{m*}$$
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Furthermore,

$$T^2_m E \oplus_{T_m E} V^\wedge_m \tau_{mE} \to \hat{T}^2_m E$$

is an isomorphism of affine bundles over $T_m E$.

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\begin{aligned}
V^{\vee}_m \tau_{mE} &\cong \tau^*_{mE} TE \otimes_{T_m E} S^2 \mathbb{R}^{m*} \\
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$$

Furthermore,

$$T^2_m E \oplus_{T_m E} V^{\wedge}_m \tau_{mE} \to \hat{T}^2_m E$$

is an isomorphism of affine bundles over $T_m E$.

The skew-symmetric component of a semiholonomic double velocity is called its curvature.

The double structure for regular double velocities

$$\begin{tikzcd}
\mathring{T}_m \mathring{T}_m E \supset \mathring{T}_m^{\mathrm{T}} \mathring{\tau}_{mE} \arrow[r, "T_m \tau_{mE}"] \arrow[d, "\tau_{m(\mathring{T}_m E)}"'] & \mathring{T}_m E \arrow[d, "\tau_{mE}"] \\
\mathring{T}_m E \arrow[r, "\tau_{mE}"'] & E
\end{tikzcd}$$

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\end{tikzcd}$$

Put

$$\mathring{\hat{T}}{}^2_m E = \hat{T}^2_m E \cap T_m \mathring{T}_m E\,, \qquad \mathring{T}^2_m E = T^2_m E \cap T_m \mathring{T}_m E$$

then

$$\mathring{T}^2_m E \subset \mathring{\hat{T}}{}^2_m E \subset \mathring{T}^{\mathrm{T}}_m \mathring{\tau}_{mE}\,.$$

The velocity group

The velocity manifold $T_m L_m$ of the jet group L_m is itself a Lie group.

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If σ_1, σ_2 are m-curves in L_m then $j_0\sigma_1, j_0\sigma_2 \in T_m L_m$, and

$$j_0\sigma_1 \cdot j_0\sigma_2 = j_0(\sigma_1 \cdot \sigma_2)$$

where $\sigma_1 \cdot \sigma_2$ is the m-curve $t \mapsto \sigma_1(t)\sigma_2(t)$.

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(This construction applies to any Lie group, and is a generalisation of the tangent group construction.)

Double velocities to double contact elements

The free right action of L_m on $\mathring{T}_m E$,

$$\alpha : L_m \times \mathring{T}_m E \to \mathring{T}_m E\,, \qquad \alpha(j_0\phi, j_0\gamma) = j_0(\gamma \circ \phi)$$

gives rise to a free right action of $T_m L_m$ on $T_m \mathring{T}_m E$,

$$\tilde{\alpha} : T_m L_m \times T_m \mathring{T}_m E \to T_m \mathring{T}_m E\,, \qquad \tilde{\alpha}(j_0\sigma, j_0\tilde{\gamma}) = j_0(\alpha \circ (\sigma, \tilde{\gamma}) \circ \Delta)$$

where σ is an m-curve in L_m, and $\Delta : \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^m$ is the diagonal inclusion map.

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The quotient $T_m \mathring{T}_m E / T_m L_m$ may be identified with the velocity manifold $T_m J_m E$.

L_m now acts freely on the open submanifold $\mathring{T}_m J_m E$, giving the quotient $J_m J_m E$.

The double group

The right action of the jet group L_m on the velocity group $T_m L_m$

$$(j_0\phi, j_0\sigma) \mapsto j_0(\sigma \circ \phi)$$

is an action by automorphisms.

The double group

The right action of the jet group L_m on the velocity group T_mL_m

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is an action by automorphisms.

Define the double group D_m to be the semidirect product $L_m \rtimes T_mL_m$.

An element of D_m is a pair $(j_0\phi, j_0\sigma)$ where ϕ is a local diffeomorphism of R^m fixing zero and σ is an m-curve in L_m.

Double m-curves and the double group

Let $\psi : O \times O \to \mathbb{R}^m$ be a double m-curve. Put

$$\tilde{\psi}_s(t) = \psi(s,t) - \psi(s,0)\,, \qquad \tilde{\psi}_t(s) = \psi(s,t) - \psi(0,t)\,.$$

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Suppose both $\tilde{\psi}_s$ and $\tilde{\psi}_t$ are local diffeomorphisms of $\mathbb{R}^m$ for all s, t, so that $j_0\tilde{\psi}_s, j_0\tilde{\psi}_t \in L_m$. Then $t \mapsto j_0\tilde{\psi}_t$ is an m-curve in L_m.

Any element $(j_0\phi, j_0\sigma) \in D_m$ may therefore be written as

$$\left(j_0\tilde{\psi}_{s=0}, j_0(t \mapsto j_0\tilde{\psi}_t)\right).$$

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Two such double m-curves ψ, χ determine the same element of D_m when

$$\frac{\partial\psi}{\partial s} = \frac{\partial\chi}{\partial s}\,, \qquad \frac{\partial\psi}{\partial t} = \frac{\partial\chi}{\partial t}\,, \qquad \frac{\partial^2\psi}{\partial s\,\partial t} = \frac{\partial^2\chi}{\partial s\,\partial t}\,.$$

Holonomic and semiholonomic subgroups

An element $(j_0\phi, j_0\sigma) \in D_m$ is holonomic if $\sigma = \tilde{j}\phi$ (so that the element is $(j_0\phi, j_0^2\phi)$).

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$\{(j_0\phi, j_0\sigma)\}$ is a closed Lie subgroup $\hat{L}_m^2 \subset D_m$ diffeomorphic to $T_m L_m$, but *not* isomorphic as a group.

The curvature subgroup

There is a map $\vee : D_m \to D_m$ given in terms of double curves by

$$\psi(s,t) \mapsto \tfrac{1}{2}(\psi(s,t) + \psi(t,s))\,.$$

This restricts to a surjective map $\hat{L}^2_m \to L^2_m$.

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There is an injective map $\mu : L_m \to \hat{L}^2_m$ given by

$$\mu(j_0\phi) = \big(j_0\phi, j_0(t \mapsto j_0\phi)\big) \, .$$

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The curvature subgroup $\tilde{L}^2_m$ is the semidirect product

$$\tilde{L}^2_m = \mu(L_m) \rtimes \ker \vee \,.$$

The double group action

$\tilde{\alpha} : T_m L_m \times T_m T_m E \to T_m T_m E\,,$
$\tilde{\alpha}(j_0\sigma, j_0\tilde{\gamma}) = j_0(\alpha \circ (\sigma, \tilde{\gamma}) \circ \Delta)$
The action is free on $T_m \mathring{T}_m E$, with quotient $T_m J_m E$.

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The action is free on $\mathring{T}_m J_m E$, with quotient $J_m J_m E$.

The double group has an action

$$\begin{aligned} \bar{\alpha} &: D_m \times T_m T_m E \to T_m T_m E \\ \bar{\alpha}((j_0\phi, j_0\sigma), j_0\tilde{\gamma}) &= \tilde{\alpha}(j_0\sigma, \alpha(j_0\phi, j_0\tilde{\gamma}))\,. \end{aligned}$$

The restriction of the action to $\mathring{T}^{\mathrm{T}}_m \rho$ is free, and the quotient may be identified with $J_m J_m E$.

The double group action

$\tilde{\alpha} : T_m L_m \times T_m T_m E \to T_m T_m E\,,$
$\tilde{\alpha}(j_0\sigma, j_0\tilde{\gamma}) = j_0(\alpha \circ (\sigma, \tilde{\gamma}) \circ \Delta)$
The action is free on $T_m \mathring{T}_m E$, with quotient $T_m J_m E$.

$\alpha : L_m \times T_m J_m E \to T_m J_m E\,, \qquad \alpha(j_0\phi, j_0\gamma) = j_0(\gamma \circ \phi)$
The action is free on $\mathring{T}_m J_m E$, with quotient $J_m J_m E$.

The double group has an action

$$\bar{\alpha} : D_m \times T_m T_m E \to T_m T_m E$$
$$\bar{\alpha}((j_0\phi, j_0\sigma), j_0\tilde{\gamma}) = \tilde{\alpha}(j_0\sigma, \alpha(j_0\phi, j_0\tilde{\gamma}))\,.$$

The restriction of the action to $\mathring{T}^{\mathrm{T}}_m \rho$ is free, and the quotient may be identified with $J_m J_m E$.
We also consider the restriction to $\mathring{T}^{\mathrm{T}}_m \mathring{\tau}_{mE} \subset \mathring{T}^{\mathrm{T}}_m \rho$, with quotient $J^{\mathrm{T}}_m \pi_{mE}$.

The double structure for contact elements

$$
\begin{tikzcd}
\mathring{T}_m\mathring{T}_mE \arrow[r, phantom, "\supset"] & \mathring{T}^{\mathrm{T}}_m\mathring{\tau}_{mE} \arrow[r, "T_m\rho"'] \arrow[rr, bend left, "T_m\tau_{mE}"] \arrow[dd, "\tau_{m(\mathring{T}_mE)}"'] & \mathring{T}^{\mathrm{T}}_m\pi_{mE} \arrow[r, "T_m\pi_{mE}"'] \arrow[d, "\hat{\rho}"'] & \mathring{T}_mE \arrow[d, "\rho"'] \arrow[dd, bend left, "\tau_{mE}"] \\
 & & J_mJ_mE \supset J^{\mathrm{T}}_m\pi_{mE} \arrow[r, "J_m\pi_{mE}"] \arrow[d, "\pi_{m(J_mE)}"'] & J_mE \arrow[d, "\pi_{mE}"'] \\
 & \mathring{T}_mE \arrow[r, "\rho"] \arrow[rr, bend right, "\tau_{mE}"'] & J_mE \arrow[r, "\pi_{mE}"] & E
\end{tikzcd}
$$

Semiholonomic contact elements

Let $\bar{\rho}$ denote the projection $\mathring{T}^{\mathrm{T}}_{m}\rho \to J_m J_m E$,
and also the restricted map $\mathring{T}^{\mathrm{T}}_{m}\mathring{\tau}_{mE} \to J^{\mathrm{T}}_{m}\pi_{mE}$.

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A contact element is semiholonomic if it is of the form
$\bar{\rho}(j_0\tilde{\gamma})$ for some $j_0\tilde{\gamma} \in \mathring{T}^{2}_{m}E$.

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The submanifold

$$\hat{J}^2_m E = \{\bar{\rho}(j_0 \tilde{\gamma}) \text{ semiholonomic}\}$$

may be identified with the quotient $\mathring{\mathring{T}}{}^2_m E / \hat{L}^2_m$.

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may be identified with the quotient $\mathring{T}^{2}_{m}E/\hat{L}^2_m$.

$\hat{J}^2_m E \to J_m E$ is an affine bundle, modelled on the vector bundle $V_m\mathring{\tau}_{mE}/\hat{L}^2_m \to J_m E$.

Holonomic contact elements

A contact element is holonomic if it is of the form $\bar{\rho}(j_0\tilde{\gamma})$ for some $j_0\tilde{\gamma} \in \mathring{T}^2_m E$.

The submanifold

$$J^2_m E = \{\bar{\rho}(j_0\tilde{\gamma}) \text{ holonomic}\}$$

may be identified with the quotient $\mathring{T}^2_m E / L^2_m$.

$J^2_m E \to J_m E$ is an affine bundle, modelled on the vector bundle $V^\vee_m \mathring{\tau}_{mE} / L^2_m \to J_m E$.

Curvature of semiholonomic contact elements

The decomposition

$$V_m \mathring{\tau}_{mE} = V_m^{\vee} \mathring{\tau}_{mE} \oplus V_m^{\wedge} \mathring{\tau}_{mE}$$

projects to a decomposition

$$V_m \mathring{\tau}_{mE} / \hat{L}_m^2 \cong V_m^{\vee} \mathring{\tau}_{mE} / L_m^2 \oplus V_m^{\wedge} \mathring{\tau}_{mE} / \tilde{L}_m^2 \,.$$

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This gives a decomposition of a semiholonomic contact element in $\hat{J}_m^2 E$

as the sum of a holonomic contact element in $J_m^2 E$ and a curvature element in $V_m^{\wedge} \mathring{\tau}_{mE} / \tilde{L}_m^2$.