Stokes' Theorem on Lie Algebroids and Some of its Applications

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R-linear forms on Lie algebroids

Let $(A, \rho_A, [\![\bullet, \bullet]\!]_A)$ be Lie algebroid over a manifold M. An $\mathbb{R}$-multilinear, antisymmetric map

$$\omega : \underbrace{\Gamma(A) \times \cdots \times \Gamma(A)}_{n} \longrightarrow \mathscr{C}^\infty(M)$$

is called an $\mathbb{R}$-*linear* n-*form* on A. The space of all such $\mathbb{R}$-linear n-forms will be denoted by $\mathcal{A}lt^n_{\mathbb{R}}(\Gamma(A); \mathscr{C}^\infty(M))$, and the space of $\mathbb{R}$-linear forms on A by

$$\mathcal{A}lt^\bullet_{\mathbb{R}}(\Gamma(A); \mathscr{C}^\infty(M)) = \bigoplus_{k \geq 0} \mathcal{A}lt^k_{\mathbb{R}}(\Gamma(A); \mathscr{C}^\infty(M)),$$

where $\mathcal{A}lt^0_{\mathbb{R}}(\Gamma(A); \mathscr{C}^\infty(M)) = \mathscr{C}^\infty(M)$.

Cohomology Space

We extend usual exterior derivative d_A on A to the exterior derivative

$$d_{A,\mathbb{R}} : \mathcal{A}lt^{\bullet}_{\mathbb{R}}\left(\Gamma\left(A\right);\mathscr{C}^{\infty}\left(M\right)\right) \longrightarrow \mathcal{A}lt^{\bullet+1}_{\mathbb{R}}\left(\Gamma\left(A\right);\mathscr{C}^{\infty}\left(M\right)\right)$$

for $\mathbb{R}$-linear forms on A by the classical formula

$$\begin{aligned}\left(d_{A,\mathbb{R}}\eta\right)\left(a_1,\ldots,a_{n+1}\right) &= \sum_{i=1}^{n+1}\left(-1\right)^{i+1}\left(\rho_A\circ a_i\right)\left(\eta\left(a_1,\ldots\widehat{a_i}\ldots,a_{n+1}\right)\right)\\ &+\sum_{i<j}\left(-1\right)^{i+j}\eta\left(\left[\!\left[a_i,a_j\right]\!\right]_A,a_1,\ldots\widehat{a_i}\ldots\widehat{a_j}\ldots,a_{n+1}\right).\end{aligned}$$

$d_{A,\mathbb{R}}$ is an antiderivation in $\mathcal{A}lt^{\bullet}_{\mathbb{R}}\left(\Gamma\left(A\right);\mathscr{C}^{\infty}\left(M\right)\right)$ with respect to the product of $\mathbb{R}$-linear forms.
The cohomology space of the complex $\left(\mathcal{A}lt^{\bullet}_{\mathbb{R}}\left(\Gamma\left(A\right);\mathscr{C}^{\infty}\left(M\right)\right),d_{A,\mathbb{R}}\right)$ we will be denote by $H_{\mathbb{R}}\left(A\right)$.

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- $$\Delta^k = \left\{ (t_1, ..., t_k) \in \mathbb{R}^k; \quad \forall i \;\; t_i \geq 0\,, \;\; \sum_{i=1}^k t_i \leq 1 \right\}$$

  the *standard* k*-simplex* in $\mathbb{R}^k$; additionally we set the *standard* 0*-simplex* as $\Delta^0 = \{0\}$.

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  the *standard* k*-simplex* in $\mathbb{R}^k$; additionally we set the *standard* 0*-simplex* as $\Delta^0 = \{0\}$.
- $\mathscr{C}^\infty(\mathbb{R} \times M)$-modules $\Gamma(\mathrm{pr}_2^* A)$ and $\mathscr{C}^\infty\left(\mathbb{R}^k \times M\right) \otimes_{\mathscr{C}^\infty(M)} \Gamma(A)$ are isomorphic and this way the module of cross-sections of the inverse image $\mathrm{pr}_2^\wedge(A) \subset T\left(\mathbb{R}^k \times M\right) \oplus \Gamma(\mathrm{pr}_2^* A)$,

  $$\mathrm{pr}_2^\wedge(A) = \left\{ (\gamma, w) \in T\left(\mathbb{R}^k \times M\right) \times A : (\mathrm{pr}_2)_* \gamma = \rho_A(w) \right\} \cong T\mathbb{R}^k \times A$$

  is a submodule of $\mathscr{X}\left(\mathbb{R}^k \times M\right) \times \left(\mathscr{C}^\infty\left(\mathbb{R}^k \times M\right) \otimes_{\mathscr{C}^\infty(M)} \Gamma(A)\right)$.

Generalized Stokes's Formula on Lie algebroids

We denote cross-sections $0 \times a$, $\frac{\partial}{\partial t^j} \times 0$ of the vector bundle $T\mathbb{R}^k \times A$ briefly by a and $\frac{\partial}{\partial t^j}$, respectively. We define the following operator

$$\int_{\Delta^k} : \mathcal{A}lt^{\bullet}_{\mathbb{R}} \left(\Gamma\left(T\mathbb{R}^k \times A\right); \mathscr{C}^{\infty}\left(\mathbb{R}^k \times M\right)\right) \longrightarrow \mathcal{A}lt^{\bullet - k}_{\mathbb{R}} \left(\Gamma\left(A\right); \mathscr{C}^{\infty}\left(M\right)\right),$$

$$\begin{aligned}
&\left(\int_{\Delta^k} \omega\right)(a_1, \ldots, a_{n-k}) \\
&= \int_{\Delta^k} \omega\left(\frac{\partial}{\partial t^1}, \ldots, \frac{\partial}{\partial t^k}, a_1, \ldots, a_{n-k}\right)_{|(t_1,\ldots,t_k,\bullet)} dt_1 \ldots dt_k
\end{aligned}$$

for all $n \geq 1$, $1 \leq k \leq n$, $\omega \in \mathcal{A}lt^{n}_{\mathbb{R}} \left(\Gamma\left(T\mathbb{R}^k \times A\right); \mathscr{C}^{\infty}\left(\mathbb{R}^k \times M\right)\right)$, $a_1,\ldots,a_{n-k} \in \Gamma(A)$ and

$$\left(\int_{\Delta^0} \omega\right)(a_1, \ldots, a_n) = \iota_0^{*}\left(\omega\left(0 \times a_1, \ldots, 0 \times a_n\right)\right), \qquad \int_{\Delta^0} f = \iota_0^{*} f$$

for $n \geq 1$, $\omega \in \mathcal{A}lt^{n}_{\mathbb{R}} \left(\Gamma\left(T\mathbb{R}^k \times A\right); \mathscr{C}^{\infty}\left(M\right)\right)$, $a_1, \ldots, a_n \in \Gamma(A)$, $f \in \mathscr{C}^{\infty}\left(\mathbb{R}^k \times M\right)$ and where $\iota_0 : M \to \Delta^0 \times M$, $\iota_0(x) = (0, x)$.

We have the following Stokes' formula for $\mathbb{R}$-linear forms on A Let k be a natural number,

$$\sigma_j^k : \mathbb{R}^k \to \mathbb{R}^{k+1}, \qquad 0 \le j \le k+1$$

are functions defined by ($t = (t_1, ..., t_k) \in \mathbb{R}^k$):

$$\begin{aligned} \sigma_0^0(0) &= 1, \quad \sigma_1^0(0) = 0, \\ \sigma_0^k(t_1, \ldots, t_k) &= \left(1 - \sum_{i=1}^k t_i, t_1, \ldots, t_k\right), \\ \sigma_j^k(t_1, \ldots, t_k) &= (t_1, \ldots, t_{j-1}, 0, t_j, \ldots, t_k), \;\; 1 \le j \le k+1. \end{aligned}$$

Then

$$\int_{\Delta^k} \circ\, d_{T\mathbb{R}^k \times A,\mathbb{R}} + (-1)^{k+1}\, d_{A,\mathbb{R}} \circ \int_{\Delta^k}$$
$$= \sum_{j=0}^{k} (-1)^j \int_{\Delta^{k-1}} \circ \left(d\sigma_j^{k-1} \times \mathrm{id}_A \right)^*, \qquad (1)$$

where $\left(\left(\int_{\Delta^{k-1}} \circ \left(d\sigma_j^{k-1} \times \mathrm{id}_A \right)^*\right) \omega\right)(a_1, ..., a_{n-k+1})$ is, by definition, equal to

$$\int_{\Delta^{k-1}} \omega \left(d\sigma_j^{k-1} \left(\frac{\partial}{\partial t^1} \right), ..., d\sigma_j^{k-1} \left(\frac{\partial}{\partial t^{k-1}} \right), a_1, ..., a_{n-k+1} \right)_{|(t_1,...,t_{k-1},\bullet)} dt_1 ... dt_{k-1}$$

and

$$\left(\left(\int_{\Delta^0} \circ \left(d\sigma_j^0 \times \mathrm{id}_A \right)^*\right) \omega\right)(a_1, ..., a_n) = \left(\sigma_j^0 \times \mathrm{id}_M \circ \iota_0 \right)^* \left(\omega \left(a_1, ..., a_n \right) \right)$$

if $k \geq 2$, $\omega \in \mathcal{A}lt^n_{\mathbb{R}}\left(\Gamma\left(T\mathbb{R}^k \times A\right); \mathscr{C}^\infty\left(\mathbb{R}^k \times M\right)\right)$, $a_i \in \Gamma(A)$, $j \in \{0, 1\}$.

If we restrict the discussion to differential (linear) forms on the right side of equality of the above Stokes formula we obtain operators of pullback of forms.

Let

$$\widetilde{\int}_{\Delta^k} = \int_{\Delta^k}\Big|\Omega\left(T\mathbb{R}^k \times A\right) : \Omega\left(T\mathbb{R}^k \times A\right) \longrightarrow \Omega\left(A\right)$$

be a restriction of $\int_{\Delta^k}$ to the module $\Omega\left(T\mathbb{R}^k \times A\right)$ of differential forms on the Lie algebroid $T\mathbb{R}^k \times A$. Therefore, as a corollary we obtain the Stokes theorem for differential forms on Lie algebroids:

Homotopy operators

Theorem

(The Stokes theorem for differential forms on Lie algebroids) For every $k \in \mathbb{N}$,

$$\widetilde{\int}_{\Delta^k} \circ d_{T\mathbb{R}^k \times A} + (-1)^{k+1} d_A \circ \widetilde{\int}_{\Delta^k} = \sum_{j=0}^{k} (-1)^j \widetilde{\int}_{\Delta^{k-1}} \circ \left(d\sigma_j^{k-1} \times \mathrm{id}_A \right)^*,$$

where $\sigma_j^k : \mathbb{R}^k \to \mathbb{R}^{k+1}$ *for* $0 \leq j \leq k$ *are functions defined above (faces) and* $\left(d\sigma_j^{k-1} \times \mathrm{id}_A \right)^* : \Omega\left(T\mathbb{R}^k \times A\right) \to \Omega\left(T\mathbb{R}^{k-1} \times A\right)$ *is the pullback of forms via the homomorphism of Lie algebroids* $d\sigma_j^{k-1} \times \mathrm{id}_A : T\mathbb{R}^{k-1} \times A \to T\mathbb{R}^k \times A$ *over* $\sigma_j^{k-1} \times \mathrm{id}_M$.

See also the analog formula by I. Vaisman in *Characteristic Classes of Lie Algebroid Morphisms*, Differential Geom. Appl. **28** (2010) 635–647.

Homotopy operators

Definition

Let $(A, \rho_A, [\![\cdot,\cdot]\!]_A)$ and $(B, \rho_B, [\![\cdot,\cdot]\!]_B)$ be Lie algebroids on manifolds M and N, respectively. A *homotopy joining two homomorphisms* $\Phi_0 : A \to B$, $\Phi_1 : A \to B$ of Lie algebroids is a homomorphism of Lie algebroids

$$\Phi : T\mathbb{R} \times A \longrightarrow B$$

with $\Phi(\theta_0, \cdot) = \Phi_0$ and $\Phi(\theta_1, \cdot) = \Phi_1$, where $\theta_0 \in T_0\mathbb{R}$ and $\theta_1 \in T_1\mathbb{R}$ are null vectors; $T\mathbb{R} \times A$ denotes the Cartesian product of Lie algebroids $T\mathbb{R}$ and A. If there exists a homotopy joining two homomorphisms, we say that these homomorphisms are *homotopic*.

See: J. Kubarski, Invariant cohomology of regular Lie algebroids, in: *Analysis and Geometry in Foliated Manifolds* (Proceedings of the VII International Colloquium on Differential Geometry, Santiago de Compostella, Spain, 26–30 July 1994), 137–151, World Sci. Publ., 1995.

Homotopy operators

Let $\Phi_0 : A \to B$, $\Phi_1 : A \to B$ be two homomorphisms of Lie algebroids and let $\Phi : T\mathbb{R} \times A \to B$ be a homotopy joining Φ_0 to Φ_1.

- Define the operator $h = \int_{\Delta^1} \circ \Phi^* : \Omega(B) \longrightarrow \Omega(A)$.

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- Define the operator $h = \int_{\Delta^1} \circ \Phi^* : \Omega(B) \longrightarrow \Omega(A)$.
- We can observe that (for $j \in \{1, 2\}$) $\int_{\Delta^0} \circ \left(d\sigma_j^0 \times \mathrm{id}_A \right)^* \circ \Phi^* = \Phi_{1-j}^*$.

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- From the above, the **Stokes formula** ($k = 1$) and the **commutativity of a pullback** of differential forms on the Lie algebroid via a homomorphism of Lie algebroids **with differentials**, we conclude that:

$$\begin{aligned} & \Phi_1^* - \Phi_0^* \\ = \; & \left(\int_{\Delta^1} \circ d_{T\mathbb{R}\times A} + (-1)^{1+1} d_A \circ \int_{\Delta^1} \right) \circ \Phi^* \\ = \; & \left(\int_{\Delta^1} \circ \Phi^* \right) \circ d_B + d_A \circ \left(\int_{\Delta^1} \circ \Phi^* \right) \\ = \; & h \circ d_B + d_A \circ h. \end{aligned}$$

Hence $h = \int_{\Delta^1} \circ \Phi^*$ is a chain homotopy operator joining Φ_0^* to Φ_1^*.

Connections

- Let $(A, \rho_A, [\![\bullet, \bullet]\!]_A)$ and $(B, \rho_B, [\![\bullet, \bullet]\!]_B)$ be Lie algebroids over the same manifold M. A homomorphism $\nabla : A \to B$ of vector bundles is called an A-*connection* in B if $\rho_B \circ \nabla = \rho_A$.
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  If an A-connection ∇ in B is a homomorphism of Lie algebroids (∇ preserves the Lie brackets) we say that ∇ is *flat*.
- By an $\mathbb{R}$-*linear connection* of A in B we called an $\mathbb{R}$-linear operator

$$\nabla : \Gamma(A) \to \Gamma(B)$$

  such that

$$\operatorname{Sec} \rho_B \circ \nabla = \operatorname{Sec} \rho_A,$$

  where $\operatorname{Sec} \rho_A : \Gamma(A) \to \mathscr{X}(M)$, $(\operatorname{Sec} \rho_A)(a) = \rho_A \circ a$.
  An $\mathbb{R}$-linear connection of A in the Lie algebroid $A(E)$ of a vector bundle E is called the $\mathbb{R}$-*linear connection of* A *on the vector bundle* E. $\mathscr{CDO}(E) = \Gamma(A(E))$.

Connections

- We call the map

$$R^{\nabla}:\Gamma(A)\times\Gamma(A)\to\Gamma(B),\quad R^{\nabla}(\alpha,\beta)=[\![\nabla_{\alpha},\nabla_{\beta}]\!]_{B}-\nabla_{[\![\alpha,\beta]\!]_{A}}$$

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a *curvature* of ∇. We see that $\nabla : \Gamma(A) \to \Gamma(B)$ is flat if $R^{\nabla} = 0$.
- Examples: • For every Lie algebroid A, the adjoint connection $\mathrm{ad} : \Gamma(A) \to \mathscr{CDO}(A)$, $\mathrm{ad}(a) = [\![a, \bullet]\!]_A$ is an $\mathbb{R}$-linear connection of A on A.

  • Connections up to homotopy on super-vector bundles (Evens, Lu, Weinstein, Crainic).

  • Non-linear connections (Crainic, Fernandes); (such connections have a local property).

  • Connections induced by a connection and an 1-form. Observe that for a Lie algebroid $(A, \rho_A, [\![\cdot,\cdot]\!]_A)$ over a compact orientable manifold M with a volume form Ω every form $\eta \in \Omega^n(A)$ on A defines an $\mathbb{R}$-linear form
  $\widetilde{\eta} \in \mathcal{A}lt^n_{\mathbb{R}}(\Gamma(A); C^\infty(M))$, $\widetilde{\eta}(a_1, \ldots, a_n) = \int_M \eta(a_1, \ldots, a_n)\,\Omega$, $a_1, \ldots, a_n \in \Gamma(A)$, which is, in general, nonlocal.

The Chern-Simons transgression forms on Lie algebroids

Let

- E be a vector bundle over M,
- $\nabla : \Gamma(A) \to \mathscr{CDO}(E)$ — an $\mathbb{R}$-linear connection of A on E.

The standard fibrewise trace $\mathrm{Tr} : \Gamma(\mathrm{End}\,E) \to \mathscr{C}^{\infty}(M)$ on $\mathrm{End}(E)$ induces a trace

$$\mathrm{Tr}_{*} : \mathcal{A}lt^{\bullet}_{\mathbb{R}}(\Gamma(A);\Gamma(\mathrm{End}\,E)) \longrightarrow \mathcal{A}lt^{\bullet}_{\mathbb{R}}(\Gamma(A);\mathscr{C}^{\infty}(M))$$

such that

$$\mathrm{Tr}_{*}(\omega)(a_1,\ldots,a_n) = \mathrm{Tr}((\omega)(a_1,\ldots,a_n)).$$

Theorem

$d_{A,\mathbb{R}} \circ \mathrm{Tr}_{*} = \mathrm{Tr}_{*} \circ d_{\mathbb{R}}^{\overline{\nabla}}$ *where* $\overline{\nabla} : \Gamma(A) \to \mathscr{CDO}(\mathrm{End}\,E)$, $\overline{\nabla}_a = [\nabla_a, \bullet]$.

The Chern character form

We set (for $p \geq 1$) $\operatorname{ch}_p(\nabla) = \operatorname{Tr}_* \left(R^{\nabla}\right)^p \in \mathcal{A}lt^{2p}_{\mathbb{R}}\left(\Gamma(A); \mathscr{C}^{\infty}(M)\right)$ where $\left(R^{\nabla}\right)^p \in \mathcal{A}lt^{2p}_{\mathbb{R}}\left(\Gamma(A); \Gamma(\operatorname{End} E)\right)$ is, for $a_1, ..., a_{2p} \in \Gamma(A)$, given by

$$\left(R^{\nabla}\right)^p (a_1, ..., a_{2p}) = \frac{1}{2^p} \sum\nolimits_{\tau \in S_{2p}} \operatorname{sgn} \tau \cdot R^{\nabla}_{a_{\tau(1)}, a_{\tau(2)}} \circ \cdots \circ R^{\nabla}_{a_{\tau(2p-1)}, a_{\tau(2p)}}.$$

The $2p$-form $\operatorname{ch}_p(\nabla)$ is called the *Chern character form* associated to ∇.
There exists exactly one $\mathbb{R}$-linear connection

$$\widetilde{\nabla} : \Gamma\left(T\mathbb{R}^k \times A\right) \longrightarrow \mathscr{CDO}\left(\operatorname{pr}_2^* E\right)$$

of $T\mathbb{R}^k \times A$ on $\operatorname{pr}_2^* E$ such that

$$\left(\widetilde{\nabla}_{\left(X, \sum_i r^i \otimes a^i\right)} (\nu \circ \operatorname{pr}_2)\right)(t, \bullet) = \nabla_{\sum_i r^i(t,\bullet) \cdot a^i} (\nu)$$

for all $\left(X, \sum_i r^i \otimes a^i\right) \in \mathscr{X}\left(\mathbb{R}^k \times M\right) \times \left(\mathscr{C}^{\infty}\left(\mathbb{R}^k \times M\right) \otimes_{\mathscr{C}^{\infty}(M)} \Gamma(A)\right)$, $\nu \in \Gamma(E)$, $t = (t_1, ..., t_k) \in \mathbb{R}^k$. The connection $\widetilde{\nabla}$ is called the *lifting* of ∇ to $T\mathbb{R}^k \times A$.

The Chern-Simons forms

Let $\nabla^0, \nabla^1, \ldots, \nabla^k : \Gamma(A) \to \mathscr{CDO}(E)$ be $\mathbb{R}$-linear connections of a Lie algebroid A on a vector bundle E and $\widetilde{\nabla}^0, \widetilde{\nabla}^1, \ldots, \widetilde{\nabla}^k : \Gamma\left(T\mathbb{R}^k \times A\right) \to \mathscr{CDO}\left(\mathrm{pr}_2^* E\right)$ be their liftings to $T\mathbb{R}^k \times A$. Then there exists an $\mathbb{R}$-linear connection

$$\nabla^{\mathrm{aff}_k} : \Gamma\left(T\mathbb{R}^k \times A\right) \longrightarrow \mathscr{CDO}\left(\mathrm{pr}_2^* E\right),$$

called the *affine combination of connections* $\widetilde{\nabla}^0, \widetilde{\nabla}^1, \ldots, \widetilde{\nabla}^k$, given by

$$\nabla^{\mathrm{aff}_k} = \left(1 - \sum_{j=1}^k t_j\right) \cdot \widetilde{\nabla}^0 + \sum_{j=1}^k t_j \cdot \widetilde{\nabla}^j.$$

The Chern-Simons forms

For all $0 < k \leq 2p$ we define an $\mathbb{R}$-linear form

$$\mathrm{cs}_p\left(\nabla^0, ..., \nabla^k\right) = \int_{\Delta^k} \mathrm{ch}_p\left(\nabla^{\mathrm{aff}_k}\right) \in \mathcal{A}lt_{\mathbb{R}}^{2p-k}\left(\Gamma\left(A\right); \mathscr{C}^\infty\left(M\right)\right)$$

called the *Chern-Simons form* for $\left(\nabla^0, ..., \nabla^k\right)$ and additionally we put $cs_p\left(\nabla^0\right) = \mathrm{ch}_p\left(\nabla^0\right)$.

Theorem

*(**The Chern-Simons formula for Lie algebroids and $\mathbb{R}$-linear connections**) Let $(A, \rho_A, [\![\cdot,\cdot]\!])$ be a Lie algebroid on a manifold M, E a vector bundle over M, $k \in \mathbb{N}$, ∇^0, ..., $\nabla^k : \Gamma(A) \to \mathscr{CDO}(E)$ $\mathbb{R}$-linear connections of A on E. Then*

$$(-1)^{k+1}\, d_{A,\mathbb{R}}\left(\mathrm{cs}_p\left(\nabla^0, ..., \nabla^k\right)\right) = \sum_{j=0}^{k} (-1)^j\, \mathrm{cs}_p\left(\nabla^0, ... \widehat{\nabla^j} ..., \nabla^k\right) \tag{2}$$

for all integer numbers p such that $0 < k \leq 2p$ and $d_{A,\mathbb{R}}\left(cs_p\left(\nabla^0\right)\right) = 0$.

In the proof we use Stoke's formula, the Bianchi identity and some technical properties. If $\nabla^0, \nabla^1, \ldots, \nabla^k : \Gamma(A) \to \mathscr{CDO}(E)$ are $\mathscr{C}^\infty(M)$-linear connections, then ∇^{aff_k} is a $\mathscr{C}^\infty(M)$-linear connection. In this case, we obtain a formula due to property of Chern-Simons transgressions in by M. Crainic and R. L. Fernandes [Secondary Characteristic Classes of Lie Algebroids. In: Quantum Field Theory and Noncommutative Geometry, Lecture Notes in Phys. 662, pp. 157–176, Springer, Berlin, 2005].

The Chern character

Let $\nabla : \Gamma(A) \to \mathscr{CDO}(E)$ be an $\mathbb{R}$-linear connection of A on a vector bundle E. According to the Chern-Simons formula, we have

$$d_{A,\mathbb{R}}\left(\mathrm{cs}_p\left(\nabla^0, \nabla^1\right)\right) = \mathrm{cs}_p\left(\nabla^1\right) - \mathrm{cs}_p\left(\nabla^0\right).$$

The Chern character forms $\mathrm{ch}_p(\nabla) \in \mathcal{A}lt^{2p}_{\mathbb{R}}\left(\Gamma(A); \mathscr{C}^\infty(M)\right)$ are closed and their cohomology classes

$$\mathrm{ch}_p(A, E) = [\mathrm{ch}_p(\nabla)] \in H^{2p}_{\mathbb{R}}(A),$$

do not depend on the choice of the connection ∇. In this way we have correctly defined the Chern character

$$\mathrm{ch}(A, E) \in H_{\mathbb{R}}(A).$$

Let E be a vector bundle over M with a metric h and $\nabla : \Gamma(A) \to \mathscr{CDO}(E)$ be an $\mathbb{R}$-linear connection of a Lie algebroid A on E.

We define an $\mathbb{R}$-linear connection

$$\nabla^h : \Gamma(A) \to \mathscr{CDO}(E)$$

of A on E such that

$$(\rho_A \circ a)(h(s,t)) = h(\nabla_a s, t) + h\left(s, \nabla^h_a t\right), \quad a \in \Gamma(A),\ s,t \in \Gamma(E).$$

We can observe that

$$R^{\nabla^h}_{a,b} = -\left(R^{\nabla}_{a,b}\right)^*, \qquad a,b \in \Gamma(A),$$

where $\left(R^{\nabla}_{a,b}\right)^*$ is the adjoint map to $R^{\nabla}_{a,b}$ with respect to h. In particular, we see that ∇^h is flat if ∇ is flat.

Secondary characteristic classes for R-linear connections

Therefore we obtain the following lemma.

Lemma

If ∇_0, ∇_1 *are* $\mathbb{R}$*-linear connections of* A *on* E*, then*

$$\mathrm{cs}_p\left(\nabla_0^h, \nabla_1^h\right) = (-1)^p\,\mathrm{cs}_p\left(\nabla_0, \nabla_1\right).$$

Assume additionally that ∇ is flat. From the Chern-Simons formula we deduce that

$$\begin{aligned} d_{A,\mathbb{R}}\,\mathrm{cs}_p\left(\nabla, \nabla^h\right) &= \mathrm{cs}_p\left(\nabla\right) - \mathrm{cs}_p\left(\nabla^h\right) \\ &= 0, \end{aligned}$$

because ∇ and ∇^h are flat.

Theorem

The cohomology class $\left[\mathrm{cs}_p\left(\nabla, \nabla^h\right)\right] \in H_{\mathbb{R}}^{2p-1}(A)$ *do not depend on the choice of metric* h*.*

Proof.

Let h_1, h_2 be two metrics on E and let ∇^M be any TM-connection on E. Thus $\nabla_o = \nabla^M \circ \rho_A$ is an A-connection on E (i.e. a linear connection). The Chern-Simons formula (2) yields

$$d_{A,\mathbb{R}}\,\mathrm{cs}_p\left(\nabla, \nabla^{h_j}, \nabla_o^{h_j}\right) = \mathrm{cs}_p\left(\nabla, \nabla_o^{h_j}\right) - \mathrm{cs}_p\left(\nabla^{h_j}, \nabla_o^{h_j}\right) - \mathrm{cs}_p\left(\nabla, \nabla^{h_j}\right)$$

$$d_{A,\mathbb{R}}\,\mathrm{cs}_p\left(\nabla, \nabla_o, \nabla_o^{h_j}\right) = \mathrm{cs}_p\left(\nabla, \nabla_o^{h_j}\right) - \mathrm{cs}_p\left(\nabla_o, \nabla_o^{h_j}\right) - \mathrm{cs}_p\left(\nabla, \nabla_o\right)$$

for $j \in \{1, 2\}$. $\mathrm{cs}_p\left(\nabla_o, \nabla_o^{h_1}\right) - \mathrm{cs}_p\left(\nabla_o, \nabla_o^{h_2}\right)$ is an exact form [see: Crainic, Fernandez: *Secondary characteristic classes*]. From this and the above lemma ($\mathrm{cs}_p\left(\nabla^{h_j}, \nabla_o^{h_j}\right) = (-1)^p\,\mathrm{cs}_p\left(\nabla, \nabla_o\right)$), we get that cohomology classes of $\mathrm{cs}_p\left(\nabla, \nabla^{h_1}\right)$ and $\mathrm{cs}_p\left(\nabla, \nabla^{h_2}\right)$ are both equal. □

Secondary characteristic classes for R-linear connections

Definition

We call

$$\mathrm{u}_{2p-1}(A,E) = \left[\mathrm{cs}_p\left(\nabla, \nabla^h\right)\right] \in H^{2p-1}_{\mathbb{R}}(A), \quad p \in \{1,\ldots,\operatorname{rank} E\},$$

the **secondary characteristic classes** of an $\mathbb{R}$-linear connection $\nabla : \Gamma(A) \to \mathscr{CDO}(E)$.

If there exists in E an invariant metric h with respect to ∇, then $\nabla^h = \nabla$ and classes $\mathrm{u}_{2p-1}(A,E)$ are equal to zero. Hence these classes are obstructions to the existence of an invariant metric with respect to ∇.

Using the Chern-Simons formula we get the following

Theorem

Let ∇, ∇_m be $\mathbb{R}$-linear connections of A on E and ∇_m be additionally metric.

(a) *If p is even, then $\mathrm{u}_{2p-1}(A, E) = 0$.*

(b) *If p is odd, then $\mathrm{cs}_p(\nabla, \nabla_m)$ is a closed form and*

$$\mathrm{u}_{2p-1}(A, E) = [2\,\mathrm{cs}_p(\nabla, \nabla_m)]\,.$$

Proof.

Let ∇_m be metric connection with respect to a metric h. On account of the Chern-Simons formula, we have

$$-d_{A,\mathbb{R}}\, \mathrm{cs}_p\left(\nabla, \nabla^h, \nabla_m\right) = \mathrm{cs}_p\left(\nabla^h, \nabla_m\right) - \mathrm{cs}_p\left(\nabla, \nabla_m\right) + \mathrm{cs}_p\left(\nabla, \nabla^h\right).$$

Now Lemma 5 leads to $\mathrm{cs}_p\left(\nabla^h, \nabla_m\right) = (-1)^p\, \mathrm{cs}_p\left(\nabla, \nabla_m\right)$, because $\nabla_m^h = \nabla_m$. It follows that

$$\mathrm{cs}_p\left(\nabla, \nabla^h\right) = \left(1 + (-1)^{p+1}\right) \mathrm{cs}_p\left(\nabla, \nabla_m\right) - d_{A,\mathbb{R}}\, \mathrm{cs}_p\left(\nabla, \nabla^h, \nabla_m\right),$$

which completes the proof. □

Thank you!

Appendix: Covariant Differential Operators

Let A be a Lie algebroid over a manifold M, E be a vector bundle over M and $\mathrm{A}(E)$ a Lie algebroid of E.
We recall that the module $\mathscr{CDO}(E)$ of sections of the Lie algebroid $\mathrm{A}(E)$ of a vector bundle E is the space of all covariant differential operators in E, i.e. $\mathbb{R}$-linear operators $\ell : \Gamma(E) \to \Gamma(E)$ such that there exists exactly one $\widetilde{\ell} \in \mathscr{X}(M)$ with $\ell(f\zeta) = f\ell(\zeta) + \widetilde{\ell}(f)\zeta$ for all $f \in \mathscr{C}^{\infty}(M)$ and $\zeta \in \Gamma(E)$; see for example:

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- K. C. H. Mackenzie, *General Theory of Lie Groupoids and Lie Algebroids*, London Math. Soc. Lecture Note Ser. 213, Cambridge Univ. Press, 2005.;
- J. Kubarski, *The Chern-Weil homomorphism of regular Lie algebroids*, Publ. Dép. Math., Nouv. Sér., Univ. Claude Bernard, Lyon, 1991, 1–69..

Appendix: Remark about connections on Lie algebroids

The notion of an A-connection in a Lie algebroid B generalizes the known notions of connections (for example usual and partial covariant derivatives in vector bundles, a connection in principal bundles, a connection in extensions of Lie algebroids). In the case where $A = TM$ and $B = \mathrm{A}(E)$ is an algebroid of a vector bundle E, TM-connections in $\mathrm{A}(E)$ are one–to–one with covariant derivatives in E. For an arbitrary Lie algebroid A and $B = \mathrm{A}(E)$ we have A-connections of E considered earlier by many authors. In case $B = \mathrm{A}(P)$ is a Lie algebroid of a principal bundle P, we get A-connections in P. In Poisson geometry an especially rule have connections acting from a Lie algebroid $T^{*}M$ associated to a given Poisson structure.