Curvature construction of symplectic forms and symplectomorphism groups

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Geometry of Manifolds and Mathematical Physics 2011

joint work with Aleksy Tralle and Jarek Kędra

Notations

- $G \longrightarrow P \longrightarrow B$ – a principal fibre bundle
- $\mathcal{H}$ – a horizontal distribution defining connection
- θ, Ω – the connection form and the curvature form
- $\langle\, , \rangle$ – the natural pairing between $\mathfrak{g}$ and its dual $\mathfrak{g}^*$
- Ω is a 2-form with values in $\mathfrak{g}$
- $\langle X, u\rangle = u(X)$ for all $X \in \mathfrak{g}$ and $u \in \mathfrak{g}^*$

Fat vectors

Definition

A vector $u \in \mathfrak{g}^*$ is fat, if the 2-form

$$(X, Y) \longmapsto \langle \Omega(X, Y), u \rangle$$

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Remark

If $u \in \mathfrak{g}^*$ is a fat vector, then its whole coadjoint orbit $\mathcal{O}(u)$ is fat.

Symplectically fat bundles

Theorem 1. (Sternberg, Weinstein)

Let there be given a principal fibre bundle

$$G \longrightarrow P \longrightarrow B$$

and a symplectic G-manifold F with a hamiltonian G-action and a moment map $\mu : F \to \mathfrak{g}^*$. If there exist a connection in the above principal bundle such that all vectors in $\mu(F) \subset \mathfrak{g}^*$ are fat, then the total space of the associated bundle

$$F \longrightarrow P \times_G F \longrightarrow B$$

admits a fiberwise symplectic structure.

Symplectically fat bundles

Proposition 1.

Let u be a fat vector in a principal fibre bundle

$$G \longrightarrow P \longrightarrow B.$$

Then the associated bundle

$$\mathcal{O}(u) \longrightarrow P \times_G \mathcal{O}(u) \longrightarrow B$$

is a symplectically fat bundle.

Notations

- G – a semisimple Lie group with a Lie algebra $\mathfrak{g}$
- $H \subset G$ – a maximal rank compact Lie subgroup
- B – the Killing form for G, which is nondegenerate on $\mathfrak{h} \subset \mathfrak{g}$
- $\mathfrak{g}^c$, $\mathfrak{h}^c$ – complexifications of the Lie algebra $\mathfrak{g}$ and $\mathfrak{h}$
- $\mathfrak{t}$ – a maximal abelian subalgebra in $\mathfrak{h}$
- $\mathfrak{t}^c$ – a Cartan subalgebra in $\mathfrak{g}^c$
- $\Delta = \Delta(\mathfrak{g}^c, \mathfrak{t}^c)$ – the root system for $\mathfrak{g}^c$ with respect to $\mathfrak{t}^c$
- $\Delta(\mathfrak{h})$ – the root system for $\mathfrak{h}^c$ with respect to $\mathfrak{t}^c$
- $\Delta(\mathfrak{h})$ is a subsystem of Δ

Notations

- $\mathfrak{m}$ – the orthogonal complement to $\mathfrak{h}$ in $\mathfrak{g}$ with respect to the Killing form B
- the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ complexifies to $\mathfrak{g}^c = \mathfrak{h}^c \oplus \mathfrak{m}^c$
- we have following root decompositions

$$\mathfrak{g}^c = \mathfrak{t}^c + \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha$$

$$\mathfrak{h}^c = \mathfrak{t}^c + \sum_{\alpha \in \Delta(\mathfrak{h})} \mathfrak{g}^\alpha$$

$$\mathfrak{m}^c = \sum_{\alpha \in \Delta \setminus \Delta(\mathfrak{h})} \mathfrak{g}^\alpha$$

Notations

- the Killing form B for G is nondegenerate (since G is semisimple)
- we can identify Lie algebra $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ via the Killing form B

$$\forall_{u \in \mathfrak{g}^*} \; u \longmapsto X_u$$

- this identification preserves the identification of $\mathfrak{h}$ and $\mathfrak{h}^*$ (since B is by assumption nondegenerate on $\mathfrak{h}$)
- $C \subset \mathfrak{t}$, C_α – a closed Weyl chamber and its wall determined by the root α

Generalisation of Lerman's Theorem

Theorem 2.

Let G be a semisimple Lie group, and $H \subset G$ a compact subgroup of maximal rank. Suppose that the Killing form B of G is nondegenerate on the Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ of the subgroup H.
The following conditions are equivalent:

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1. A vector $u \in \mathfrak{h}^*$ is fat with respect to the cannonical invariant connection in the principal bundle $H \longrightarrow G \longrightarrow G/H$.

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$$Ad_H(\cup_{\alpha \in \Delta \setminus \Delta(\mathfrak{h})} C_\alpha).$$

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2. The vector X_u does not belong to the set
$$Ad_H(\cup_{\alpha \in \Delta \setminus \Delta(\mathfrak{h})} C_\alpha).$$
3. The isotropy subgroup $V \subset H$ of $u \in \mathfrak{h}^*$ with respect to the coadjoint action is the centralizer of a torus in G.

Generalisation of Lerman's Theorem

Corollary

Let G be a semisimple Lie group of noncompact type, which is a real form of a semisimple complex Lie group.
Assume that H is a maximal compact subgroup in G.
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Definition

We will call the vector $u \in \mathfrak{h}^*$ an admissible vector, if and only if its dual vector $X_u \in \mathfrak{h}$ does not belong to the set

$$Ad_H(\cup_{\alpha \in \Delta \setminus \Delta(\mathfrak{h})} C_\alpha).$$

Coadjoint orbits as fibers

Theorem 3.

Let G be a semisimple Lie group and $H \subset G$ a compact subgroup. The cannonical invariant connection in the principal bundle

$$H \longrightarrow G \longrightarrow G/H$$

admits fat vectors, if $\operatorname{rank} G = \operatorname{rank} H$.
If G is compact, the converse is also true.

Coadjoint orbits as fibers

Remark.

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1. Any compact simply connected homogeneous symplectic manifold is symplectomorphic to a coadjoint orbit. However Proposition 1 is applicable only to coadjoint orbits of admissible vectors.
2. Only homogeneous spaces can have coadjoint orbits as images of the moment map. Thus there is no possibility to extend the class of symplectically fat fiber bundles in a following way: take any symplectic G-manifold and require that $\mu(F)$ is a coadjoint orbit of some fat vector.

New examples

Proposition 2.

Let G be a semisimple Lie group and $H \subset G$ its compact subgroup of maximal rank. Assume that $T \subset H$ is a maximal torus. Then the associated bundle $H/T \longrightarrow G/T \longrightarrow G/H$ is always symplectically fat.

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Proposition 3.

Let G be a semisimple Lie group and $T \subset G$ a torus and a subgroup of maximal rank. Assume that (M, ω) is a closed symplectic manifold with a Hamiltonian action of the torus T. Then the associated bundle $M \longrightarrow G \times_T M \longrightarrow G/T$ is symplectically fat.

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Proposition 4.

Any homogeneous quaternionic Kähler symmetric Riemannian manifold G/H of a nonzero Ricci curvature is a base of a symplectically fat bundle with $\mathcal{O}(u)$ as a fibre, where u is an admissible vector.

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Proposition 5.

Let G be a semisimple Lie group, $K \subset G$ its maximal compact subgroup of maximal rank and $\Gamma \subset G$ a lattice trivially intersecting K. Let $V \subset K$ be a connected subgroup that is the centralizer of a torus in G. Then associated bundles $K/V \longrightarrow G/V \longrightarrow G/K$ and $K/V \longrightarrow \Gamma \setminus G/V \longrightarrow \Gamma \setminus G/K$ are symplectically fat.

Locally homogeneous complex manifolds

Definition.

Locally homogeneous complex manifolds are manifolds of the form $\Gamma \setminus G/V$, where G is a noncompact real form of a complex semisimple Lie group G^c and $V = G \cap P$, where $P \subset G^c$ is a parabolic subgroup.

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Locally homogeneous complex manifolds are manifolds of the form $\Gamma \setminus G/V$, where G is a noncompact real form of a complex semisimple Lie group G^c and $V = G \cap P$, where $P \subset G^c$ is a parabolic subgroup.

Lemma.

Let G be a semisimple Lie group of noncompact type which is a real form of a complex semisimple Lie group G^c. Let $P \subset G^c$ be a parabolic subgroup such that $V = P \cap G$ is compact. Then $V = Z_G(S)$ is the centralizer of a torus $S \subset G$.

Locally homogeneous complex manifolds

Theorem 4.

Let G be a noncompact real form of a complex semisimple Lie group G^c and let $P \subset G^c$ be a parabolic subgroup such that $V = P \cap G$ is compact. Assume that $K \subset G$ is a maximal compact subgroup containing V. Then the associated bundle

$$K/V \longrightarrow G/V \longrightarrow G/K$$

is symplectically fat. Moreover if $\Gamma \subset G$ is a cocompact lattice trivially intersecting K, then the bundle

$$K/V \longrightarrow \Gamma \setminus G/V \longrightarrow \Gamma \setminus G/K$$

is also symplectically fat.

Twistor spaces

Definition

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Theorem 5.

The twistor bundle

$$SO(2n)/U(n) \longrightarrow SO(2n+1)/U(n) \longrightarrow S^{2n}$$

over an even dimensional sphere is symplectically fat.

Twistor spaces

Proposition 6.

The associated bundle

$$SO(2n)/U(n) \longrightarrow SO(2n,1)/U(n) \longrightarrow SO(2n,1)/SO(2n)$$

is a symplectically fat twistor bundle. Assume that Γ is a cocompact lattice trivially intersecting $SO(2n)$, then the associated bundle

$$SO(2n)/U(n) \longrightarrow \Gamma \setminus SO(2n,1)/U(n) \longrightarrow \Gamma \setminus SO(2n,1)/SO(2n)$$

is also a symplectically fat twistor bundle.

Twistor spaces

Definition

The sectional curvature K_g of a Riemannian manifold (M, g) is called ε-pinched if it satisfies the following inequality

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Theorem 6.

The orthonormal frame bundle

$$SO(2n) \longrightarrow P \longrightarrow M$$

over Riemannian manifold M with $\frac{3}{2n+1}$-pinched curvature admits fat vectors $f \in so(2n)^*$.

Sternberg's coupling form

Theorem 7. (Sternberg)

Let there be given a principal fibre bundle $G \longrightarrow P \longrightarrow B$ with the connection form θ and a symplectic G-manifold (F, ω_F) with a hamiltonian G-action and a moment map $\mu : F \to \mathfrak{g}^*$. The total space of the associated bundle $F \longrightarrow P \times_G F \longrightarrow B$ admits a closed 2-form Ω such that:

1. $i^*\Omega = \omega_F$;
2. $\tilde{\pi}^*\Omega = d < pr_1^*\theta, pr_2^*\mu > + pr_2^*\omega_F$;

where:

$$i : (F, \omega_F) \to P \times_G F, \qquad \tilde{\pi} : P \times F \to P \times_G F,$$

$$pr_1 : P \times F \to P, \qquad pr_2 : P \times F \to F.$$

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2. The coupling form Ω can also be expressed either in terms of the connection form θ, its curvature form Θ and symplectic form on the fiber or in terms of the fibre integration.

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Definition

The associated bundle $F \longrightarrow P \times_G F \longrightarrow B$ is a symplectically fat bundle if and only if its Sternberg's coupling form Ω is nondegenerate.

Universal bundle of hamiltonian fibration

Let $(F, \omega_F) \longrightarrow E = P \times_G F \longrightarrow B$ be a Hamiltonian bundle (G is a subgroup of $Ham(F, \omega_F)$). We have the following diagram:

$$\begin{array}{ccc} F & = & F \\ \downarrow & & \downarrow \\ E & \longrightarrow & EG \times_G F \\ \downarrow & & \downarrow \\ B & \xrightarrow[f]{} & BG \end{array}$$

where $F \longrightarrow F_G = EG \times_G F \longrightarrow BG$ is an universal bundle.

Cohomology of $H^*(B\,Ham(F, \omega_F))$

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Remark.

1. Each Hamiltonian fibre bundle $F \longrightarrow E \longrightarrow B$ together with its universal Hamiltonian fibration $F \longrightarrow F_G \longrightarrow BG$ and the classifying map $f : B \to B\,Ham(F, \omega_F)$ gives the cohomology homomorphism:

$$f^* : H^*(B\,Ham(F, \omega_F)) \to H^*(B).$$

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$$f^* : H^*(B\,Ham(F, \omega_F)) \to H^*(B).$$

2. Each new Hamiltonian fibration gives some information about $H^*(B\,Ham(F, \omega_F))$.
3. The coupling form determines certain cohomology class of the base (via the fibre integration) an can be used to study $H^*(B\,Ham(F, \omega_F))$.

References

J. Kędra, A. Tralle, A. Woike, *On nondegenerate coupling forms*, J. Geom. Phys. 61 (2011), 462-465.