Reduction of Poisson-Nijenhuis Lie algebroids

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joint work with Juan Carlos Marrero and Edith Padrón**

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Motivation

First a preliminary question:

- Why study Poisson-Nijenhuis Lie algebroids?

Poisson-Nijenhuis manifolds (briefly)

Let M be a manifold, Λ a bivector field and N a $(1,1)$-tensor on M.

- Λ is a Poisson structure, i.e., $[\Lambda, \Lambda] = 0$
- N is Nijenhuis operator, i.e., $\mathcal{T}_N = 0$
- Λ and N satisfy the compatibility conditions

$$N \circ \Lambda^\sharp = \Lambda^\sharp \circ N^*, \quad C(\Lambda, N) = 0$$

where $\Lambda^\sharp : T^*M \to TM, \quad \Lambda^\sharp(\alpha) = i_\alpha \Lambda$

$$\Downarrow$$

(M, Λ, N) Poisson-Nijenhuis manifold

Poisson-Nijenhuis manifolds

(M, Λ, N) Poisson-Nijenhuis manifold $\Rightarrow \Lambda_i^\sharp = N\Lambda_{i-1}^\sharp$

Poisson structures Λ_i, Λ_j are compatible

Particular case:

Bi-hamiltonian manifold $(M, \Lambda_0, \Lambda_1)$ with Λ_0 symplectic structure

$$\Downarrow$$

$(M, \Lambda_0, N = {\Lambda_1}^\sharp \circ (\Lambda_0^\sharp)^{-1})$ Poisson-Nijenhuis manifold

$$+$$

$X_1 = {\Lambda_1}^\sharp(dH_0) = {\Lambda_0}^\sharp(dH_1)$ bi-Hamiltonian vector field

$$\Downarrow$$

$X_i = N^{i-1}X_1$ sequence of bi-Hamiltonian v. fields

A simple Example: Toda lattice (for two particles)

$\mathbb{R}^4$ with coordinates (q^1, q^2, p_1, p_2)

$$H_1 = \frac{1}{2}(p_1^2 + p_2^2) + e^{q^1 - q^2}$$

Poisson structures

$$\Lambda_0 = \frac{\partial}{\partial q^1} \wedge \frac{\partial}{\partial p_1} + \frac{\partial}{\partial q^2} \wedge \frac{\partial}{\partial p_2}$$

$$\Lambda_1 = -\frac{\partial}{\partial q^1} \wedge \frac{\partial}{\partial q^2} + p_1 \frac{\partial}{\partial q^1} \wedge \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial q^2} \wedge \frac{\partial}{\partial p_2} + e^{q^1 - q^2} \frac{\partial}{\partial p_2} \wedge \frac{\partial}{\partial p_1}$$

$$N = \Lambda_1^\sharp \circ (\Lambda_0^\sharp)^{-1}, \quad (\mathbb{R}^4, \Lambda_0, N) \text{ PN-manifold}$$

$$X_1 = \Lambda_0^\sharp(dH_1) = \Lambda_1^\sharp(dH_0), \qquad H_0 = p_1 + p_2$$

Toda lattice in Flaschka coordinates

$$\mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}^4, \quad (t, (q^1, q^2, p_1, p_2)) \to (q^1 + t, q^2 + t, p_1, p_2)$$

$$\mathbb{R}^4/\mathbb{R} \cong (\mathbb{R}^+) \times \mathbb{R}^2$$
$$[(q^1, q^2, p_1, p_2)] \to (e^{q_1 - q_2}, p_1, p_2)$$

$$\pi : \mathbb{R}^4 \to \mathbb{R}^+ \times \mathbb{R}^2, \;\; (q^1, q^2, p_1, p_2) \to (e^{q_1 - q_2}, p_1, p_2)$$

(a, b_1, b_2) the coordinates on the reduced space $\mathbb{R}^+ \times \mathbb{R}^2$

Toda lattice in Flaschka coordinates

Poisson reduced structures

$$\bar{\Lambda}_0 = a\frac{\partial}{\partial a} \wedge \left(\frac{\partial}{\partial b_1} - \frac{\partial}{\partial b_2}\right)$$

$$\bar{\Lambda}_1 = a\frac{\partial}{\partial a} \wedge \left(b_1\frac{\partial}{\partial b_1} - b_2\frac{\partial}{\partial b_2}\right) - a\frac{\partial}{\partial b_1} \wedge \frac{\partial}{\partial b_2}$$

$$\bar{H}_1 = \frac{1}{2}(b_1^2 + b_2^2) + a, \quad \bar{H}_0 = b_1 + b_2$$

$$\bar{X}_1 = \bar{\Lambda}_0^\sharp(d\bar{H}_1) = \bar{\Lambda}_1^\sharp(d\bar{H}_0)$$

$\nexists \bar{N}$ such that $\bar{\Lambda}_1^\sharp = \bar{N} \circ \bar{\Lambda}_0^\sharp$!!!

What happens?

The answer is in the theory of Poisson-Nijenhuis Lie algebroids

Lie algebroids

Definition (Pradines, 1967)

A Lie algebroid is a vector bundle $\tau_A \colon A \to M$ endowed with

(i) an *anchor*, i.e., a vector bundle morphism $\rho_A \colon A \to TM$

(ii) a Lie algebra bracket on $\Gamma(A)$, $[\, ,\,]_A$, such that

$$[X, fY]_A = f\,[X, Y]_A + \rho_A(X)(f)Y,$$

for all $X, Y \in \Gamma(A)$, $f \in C^\infty(M)$.

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It follows that

$$\rho_A\left([X, Y]_A\right) = \left[\rho_A(X), \rho_A(Y)\right].$$

Examples of Lie algebroids

The Atiyah algebroid

$\pi : M \to M/G$ principal G-bundle

- $A = TM/G \to M/G$ sections are G-invariant vector fields
- $\rho_A([v]) = T\pi(v)$ induced projection map
- $[\, , \,]_A =$ bracket of G-invariant vector fields

Cartan calculus

Associated to a given Lie algebroid $(A, [\,,\,]_A, \rho_A)$ there is a *Lie algebroid differential* $\mathrm{d}^A \colon \Gamma(\wedge^\bullet A^*) \to \Gamma(\wedge^{\bullet+1} A^*)$ defined by

$$
\begin{aligned}
(\mathrm{d}^A\omega)(X_0,\ldots,X_k) := {} & \sum_{i=0}^{k}(-1)^i \rho_A(X_i)\left(\omega(X_0,\ldots,\hat{X}_i,\ldots,X_k)\right) \\
& + \sum_{0\leq i<j\leq k}(-1)^{i+j}\omega([X_i,X_j]_A, X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k),
\end{aligned}
$$

for $\omega \in \Gamma(\wedge^k A^*)$, $X_0,\ldots,X_k \in \Gamma(A)$.

- For $X \in \Gamma(A)$,

$$\mathcal{L}^A_X := i_X d^A + d^A i_X$$

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Morphisms of Lie algebroids

Let $(A, [\,,\,]_A, \rho_A)$ and $(A', [\,,\,]_{A'}, \rho_{A'})$ be Lie algebroids. A vector bundle morphism

$$\begin{array}{ccc} A & \xrightarrow{F} & A' \\ {\scriptstyle \tau_A}\downarrow & & \downarrow{\scriptstyle \tau_{A'}} \\ M & \xrightarrow{f} & M' \end{array}$$

is called a morphism of Lie algebroids from A to A', if

$$\mathrm{d}^A(F^*\alpha') = F^*(\mathrm{d}^{A'}\alpha') \qquad \text{for all } \alpha' \in \Gamma(\wedge^k A'^*).$$

Poisson structures on Lie algebroids

Let A be a Lie algebroid and P a section of the vector bundle $\wedge^2 A \to M$. We denote by $P^\sharp$ the usual bundle map

$$P^\sharp \colon A^* \longrightarrow A \colon \alpha \longmapsto P^\sharp(\alpha) = i_\alpha P.$$

Definition

A Poisson structure on A is a section $P \in \Gamma(\wedge^2 A)$, such that

$$[P, P]_A = 0.$$

In this case, the bracket

$$[\alpha, \beta]_P := \mathcal{L}^A_{P^\sharp\alpha}\beta - \mathcal{L}^A_{P^\sharp\beta}\alpha - \mathrm{d}^A\left(P(\alpha, \beta)\right), \quad \alpha, \beta \in \Gamma(A^*),$$

is a Lie bracket and $A^*_P = (A^*, [\,,\,]_P, \rho_A \circ P^\sharp)$ is a Lie algebroid.

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Nijenhuis operators

Let $(A, [\,,\,], \rho_A)$ be a Lie algebroid and $N : A \to A$ a vector bundle endomorphism. The torsion of N is defined by

$$\mathcal{T}_N(X, Y) := [NX, NY]_A - N[X, Y]_N, \quad X, Y \in \Gamma(A),$$

where

$[X, Y]_N := [NX, Y]_A + [X, NY]_A - N[X, Y]_A, \quad X, Y \in \Gamma(A).$

When $\mathcal{T}_N = 0$, N is called a Nijenhuis operator, $A_N = (A, [\,,\,]_N, \rho_N = \rho_A \circ N)$ is a new Lie algebroid and

$$N : A_N \to A$$

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Poisson-Nijenhuis Lie algebroids

On a Lie algebroid A with a Poisson structure $P \in \Gamma(\wedge^2 A)$, we say that a VB morphism $N : A \to A$ is compatible with P if

(i) $NP^\sharp = P^\sharp N^*$,

(ii) $[\alpha, \beta]_{NP} - [\alpha, \beta]_P^{N^*} = 0$,

Definition (Grabowski-Urbanski, 1997)

A Poisson-Nijenhuis Lie algebroid (A, P, N) is a Lie algebroid A equipped with a Poisson structure P and a Nijenhuis operator $N : A \to A$ compatible with P.

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Toda lattice in Flaschka coordinates

$$\mathbb{R}^+ \times \mathbb{R}^2 \text{ with coordinates } (a, b_1, b_2)$$

$$\pi : \mathbb{R}^4 \to \mathbb{R}^+ \times \mathbb{R}^2$$

Poisson reduced structures on $\mathbb{R}^+ \times \mathbb{R}^2$

$$\bar{\Lambda}_0 = a\frac{\partial}{\partial a} \wedge (\frac{\partial}{\partial b_1} - \frac{\partial}{\partial b_2})$$

$$\bar{\Lambda}_1 = a\frac{\partial}{\partial a} \wedge (b_1\frac{\partial}{\partial b_1} - b_2\frac{\partial}{\partial b_2}) - a\frac{\partial}{\partial b_1} \wedge \frac{\partial}{\partial b_2}$$

Toda lattice in Flaschka coordinates

$$T(\mathbb{R}^+ \times \mathbb{R}^2) \to \mathbb{R}^+ \times \mathbb{R}^2, \quad ([\cdot,\cdot], Id)$$

The Lie algebroid

$$A = \mathbb{R} \times T(\mathbb{R}^+ \times \mathbb{R}^2) \to \mathbb{R}^+ \times \mathbb{R}^2$$

$$\{e_0 = (1,0), e_1 = (0, \frac{\partial}{\partial a}), e_2 = (0, \frac{\partial}{\partial b_1}), e_3 = (0, \frac{\partial}{\partial b_2})\}$$

$$[e_i, e_j]_A = 0, \quad \rho(e_0) = 0, \quad \rho(e_1) = \frac{\partial}{\partial a}, \quad \rho(e_2) = \frac{\partial}{\partial b_1}, \quad \rho(e_3) = \frac{\partial}{\partial b_2}$$

Two Poisson structures on the Lie algebroid $A = \mathbb{R} \times T(\mathbb{R}^+ \times \mathbb{R}^2)$

$$P_0 = ae_1 \wedge (e_2 - e_3) + e_0 \wedge e_3$$

$$P_1 = ae_0 \wedge e_1 + ae_1(b_1 e_2 - b_2 e_3) + ae_2 \wedge e_3 + b_2 e_0 \wedge e_3$$

These Poisson structures induce $\bar{\Lambda}_0, \bar{\Lambda}_1$ on $\mathbb{R}^+ \times \mathbb{R}^2$.

Toda lattice in Flaschka coordinates

The Nijenhuis operator N

$$N = P_1^\sharp \circ (P_0^\sharp)^{-1} : A \to A$$

$$\Downarrow$$

The Poisson-Nijenhuis Lie algebroid

$$(A = \mathbb{R} \times T(\mathbb{R}^+ \times \mathbb{R}^2) \to \mathbb{R}^+ \times \mathbb{R}^2, P_0, N)$$

$$P_0^\sharp d^A \bar{H}_1 = P_1^\sharp d^A \bar{H}_0$$

$$\bar{\Lambda}_0^\sharp d\bar{H}_1 = \rho_A(P_0^\sharp d^A \bar{H}_1) = \rho_A(P_1^\sharp d^A \bar{H}_0) = \bar{\Lambda}_1^\sharp d\bar{H}_0$$

R. Caseiro: Modular classes of Poisson-Nijenhuis Lie algebroids, Lett. Math. Phys. **80** (2007) 223–238

Poisson-Nijenhuis Lie algebroids

General case

$\pi : M \to M/G$ principal bundle

(M, P, N) PN-manifold

P, N G-invariants

$\widetilde{\pi} : TM/G \to M/G$
Atiyah algebroid

$(\widetilde{P}, \widetilde{N})$ *PN*-Lie algebroid

M/G is not, in general,

PN-manifold!!

Toda lattice

$\pi : \mathbb{R}^4 \to \mathbb{R}^4/\mathbb{R} \cong \mathbb{R}^+ \times \mathbb{R}^2$ principal bundle

$(\mathbb{R}^4, P, N)$ PN-manifold
P, N $\mathbb{R}$-invariants

$\widetilde{\pi} : T\mathbb{R}^4/\mathbb{R} \cong \mathbb{R} \times T(\mathbb{R}^+ \times \mathbb{R}^2) \to \mathbb{R}^+ \times \mathbb{R}^2$

Atiyah algebroid
$(\widetilde{P}, \widetilde{N})$ *PN*-Lie algebroid

$\mathbb{R}^+ \times \mathbb{R}^2$ is not PN-manifold!!

Reduction of PN Lie algebroids

(A, P, N)

Poisson-Nijenhuis Lie algebroid

$\Downarrow$ Reduction by restriction

$(\bar{A}, \bar{P}, \bar{N})$

symplectic-Nijenhuis Lie algebroid

$\Downarrow$ Reduction by projection

$(\widetilde{A}, \widetilde{P}, \widetilde{N})$

symplectic-Nijenhuis Lie algebroid with $\widetilde{N}$ nondegenerate

i.e. $\widetilde{P}^\sharp : \widetilde{A}^* \to \widetilde{A}$, $\widetilde{N} : \widetilde{A} \to \widetilde{A}$ isomorphisms

1^{st} step: Reduction by restriction

$(A, [\,,\,]_A, \rho_A, P, N)$ Poisson-Nijenhuis Lie algebroid on M.

Distribution $D \subset TM$, $D(x) := \rho_A(P^\sharp(A^*_x)) \subset T_xM$ for $x \in M$

$$\left[\rho_A(P^\sharp\alpha), \rho_A(P^\sharp\beta)\right] = \rho_A(P^\sharp[\alpha,\beta]_P)$$

$+$

D is locally finitely generated

$\Downarrow$

D is a generalized foliation of M in the sense of Sussmann.

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1^{st} step: Reduction by restriction

- Let $L \subset M$ be a leaf of the foliation $D = \rho_A(P^\sharp(A^*)) \subset TM$
- Assume: $P^\sharp : A^* \to A$ has constant rank on each leaf L.

$$\downarrow$$

$A_L := P^\sharp(A^*)_{|L} \subset A \to L$ is a Lie (sub)algebroid

- $\left[P^\sharp \alpha_{|L}, P^\sharp \beta_{|L}\right]_{A_L} = P^\sharp [\alpha, \beta]_{P|L} \in \Gamma(A_L)$
- $\rho_{A_L} = (\rho_A)_{|A_L} : A_L \to TL$

1^{st} step: Reduction by restriction

$$\begin{array}{ccc} L & \xrightarrow{X_L} & A_L \\ \iota \downarrow & & \downarrow I \\ M & \xrightarrow{P^\sharp \alpha} & A \end{array}$$

$\alpha \in \Gamma(A^*)$

The symplectic structure $\Omega_L : L \to \wedge^2 A_L^*$

$$\Omega(X_L, Y_L) = P(\alpha, \beta) \circ \iota$$

Nijenhuis tensor $N_L : A_L \to A_L$

$$I \circ N_L(X_L) = N(P^\sharp \alpha) \circ \iota$$

1^{st} step: Reduction by restriction

Theorem 1

Let (A, P, N) be a Poisson-Nijenhuis Lie algebroid such that the Poisson structure has constant rank in the leaves of the foliation $D = \rho_A(P^\sharp(A^*))$. Then, we have a symplectic-Nijenhuis Lie algebroid (A_L, Ω_L, N_L) on each leaf L of D.

2^{nd} step: Reduction by projection

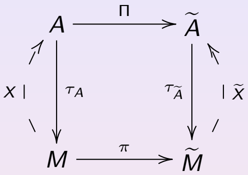
Lie algebroid epimorphism

Let $\tau_A \colon A \to M$ and $\tau_{\widetilde{A}} \colon \widetilde{A} \to \widetilde{M}$ be Lie algebroids

$$\begin{array}{ccc} A & \xrightarrow{\Pi} & \widetilde{A} \\ {\scriptstyle \tau_A}\downarrow & & \downarrow{\scriptstyle \tau_{\widetilde{A}}} \\ M & \xrightarrow{\pi} & \widetilde{M} \end{array}$$

- (Π, π) epimorphism of vector bundles
- $d^A(\Pi^*\widetilde{\alpha}) = \Pi^*(d^{\widetilde{A}}\widetilde{\alpha})$ for all $\widetilde{\alpha} \in \Gamma(\wedge^k \widetilde{A}^*)$ and all k

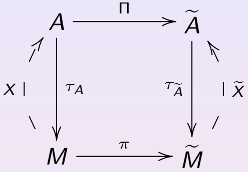
Projectability



- Π-projectable 1-section: $X \in \Gamma(A)$ such that there exists $\widetilde{X} \in \Gamma(\widetilde{A})$ with $\Pi \circ X = \widetilde{X} \circ \pi$.
- Π-projectable 2-section: $P \in \Gamma(\wedge^2 A)$ such that for all $\widetilde{\alpha} \in \Gamma(\widetilde{A}^*)$ the 1-section $P^\sharp(\Pi^* \widetilde{\alpha}) \in \Gamma(A)$ is Π-projectable.
- Π-projectable $(1,1)$-section: $N\colon A \to A$ vector bundle morphism such that

$$N(\Gamma_p(A)) \subseteq \Gamma_p(A) \quad \text{and} \quad N(\Gamma(Ker\Pi)) \subseteq \Gamma(Ker\Pi).$$

Projectability



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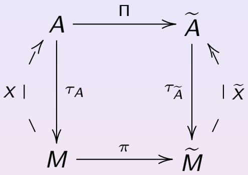

- **Π-projectable 1-section:** $X \in \Gamma(A)$ such that there exists $\widetilde{X} \in \Gamma(\widetilde{A})$ with $\Pi \circ X = \widetilde{X} \circ \pi$.
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- **Π-projectable $(1,1)$-section:** $N \colon A \to A$ vector bundle morphism such that

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Reduction by epimorphisms of Lie algebroids

$$\begin{array}{ccc} A & \xrightarrow{\Pi} & \widetilde{A} \\ \downarrow{\scriptstyle \tau_A} & & \downarrow{\scriptstyle \tau_{\widetilde{A}}} \\ M & \xrightarrow{\pi} & \widetilde{M} \end{array}$$

Theorem

Let $(\Pi, \pi) : A \to \widetilde{A}$ be a Lie algebroid epimorphism. Assume that (P, N) is a Poisson-Nijenhuis structure on A such that P and N are Π-projectable. Then, $(\widetilde{P}, \widetilde{N})$ is a Poisson-Nijenhuis structure on $\widetilde{A}$.

Complete and vertical lifts

- $(A, [\,,\,]_A\,, \rho_A)$ a Lie algebroid
- $X \in \Gamma(A)$

The vertical lift of X: $X^v \in \mathfrak{X}(A)$

(i) $X^v(f \circ \tau_A) = 0, \quad f \in \mathcal{C}^\infty(M),$

(ii) $X^v(\hat{\alpha}) = \alpha(X) \circ \tau_A, \quad \alpha \in \Gamma(A^*).$

Here, if $\alpha \in \Gamma(A^*)$ then $\hat{\alpha} \colon A \to \mathbb{R}$ is defined by

$$\hat{\alpha}(a) = \alpha(\tau_A(a))(a), \quad \text{for all } a \in A.$$

Complete and vertical lifts

- $(A, [\, , \,]_A, \rho_A)$ a Lie algebroid
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(i) $X^c(f \circ \tau_A) = \rho_A(X)(f) \circ \tau_A, \quad f \in C^\infty(M),$

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Reduction by lifts of sections of a Lie subalgebroid

Let $\tau_A \colon A \to M$ a vector bundle and $(A, [\,,\,]_A, \rho_A)$ a Lie algebroid. Consider a Lie subalgebroid $\tau_B \colon B \to M$ of A.

Key Fact

The distributions $\rho_A(B)$ and $\mathcal{F}^B$ defined by

$$\mathcal{F}^B_a := \{X^c(a) + Y^v(a) \mid X, Y \in \Gamma(B)\} \subseteq T_aA, \quad \text{for all } a \in A$$

are generalized foliations.

Now assume that

(*i*) $\rho_A(B)$ and $\mathcal{F}^B$ are regular foliations;

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We define $\tau_{\widetilde{A}} \colon \widetilde{A} = A/\mathcal{F}^B \to \widetilde{M} = M/\rho_A(B)$ such that the following diagram is commutative

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Proposition

In the above conditions we can define a Lie algebroid structure on

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The Riesz index

Let (A, P, N) a Poisson-Nijenhuis Lie algebroid. For any $x \in M$ consider the map $N_x \colon A_x \to A_x$. Recall that there exists a smallest integer $k > 0$ such that the sequences

$$\mathrm{Im}\, N_x \supseteq \mathrm{Im}\, N_x^2 \supseteq \dots$$

and

$$\ker N_x \subseteq \ker N_x^2 \subseteq \dots$$

both stabilize at rank k. That is,

$$\mathrm{Im}\, N_x^k = \mathrm{Im}\, N_x^{k+1} = \dots, \qquad \text{while } \mathrm{Im}\, N_x^{k-1} \neq \mathrm{Im}\, N_x^k,$$

and

$$\ker N_x^k = \ker N_x^{k+1} = \dots, \qquad \text{while } \ker N_x^{k-1} \neq \ker N_x^k.$$

The integer k is called the Riesz index of N at x.

The Reduced nondegenerate SN Lie algebroid

Theorem 2

Let $(A \to M, [\,,\,]_A, \rho_A, \Omega, N)$ be a symplectic-Nijenhuis Lie algebroid such that

1) N has constant Riesz index k.
2) $\rho_A(B)$ and $\mathcal{F}^B$ are regular foliations for $B = \ker N^k$.
3) For all $x \in M$, $a_x, a'_x \in L_{\mathcal{F}^B} \Longrightarrow a_x - a'_x \in \ker(N^k_x)$.

Then, we can induce a symplectic-Nijenhuis Lie algebroid structure $([\,,\,]_{\widetilde{A}}, \rho_{\widetilde{A}}, \widetilde{\Omega}, \widetilde{N})$ on $\widetilde{A} = A/\mathcal{F}^B \longrightarrow \widetilde{M} = M/\rho_A(\ker N^k)$ with $\widetilde{N}$ nondegenerate.

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Reduction:Summary

$$(A \to M, [\,,\,]_A, \rho_A, P, N)$$

Poisson-Nijenhuis Lie algebroid

$$\Downarrow \quad D = \rho_A(P^\sharp(A^*))$$

$$(A_L = P^\sharp(A^*)_{|L} \to L, [\,,\,]_{A_L}, \rho_{A_L}, \Omega_L, N_L)$$

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$$\Downarrow \quad \mathcal{F}^B = \{X^c + Y^\vee / X, Y \in \Gamma(\ker N_L^k)\}$$

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