ON THE STRUCTURE OF SOME HOMEOMORPHISM GROUPS OF MANIFOLDS

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joint work with Tomasz Rybicki

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Geometry of Manifolds and Mathematical Physics
to celebrate 80th birthday of Wlodzimierz Tulczyjew
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INTRODUCTION

Let M be a topological metrizable manifold possibly with boundary, $\dim M = n \geq 1$. Let $\mathcal{H}(M)$ be the path connected identity component of the group of all homeomorphisms of a manifold M and let $\mathcal{H}_c(M)$ be its subgroup of all elements that can be joined with the identity by a compactly supported isotopies. Both groups are endowed with the compact-open topology.

If $\partial_M \neq \emptyset$ then $M^0 = \mathrm{int} M$.
We will consider following groups:

$$\mathcal{H}_c(M^0) \leq \mathcal{H}_c^{\partial}(M) \leq \mathcal{H}_c(M) \leq \mathcal{H}(M^0).$$

Here $h \in \mathcal{H}_c^{\partial}(M)$ if there is a compactly supported isotopy h_t connecting $h_0 = \mathrm{id}$ with $h_1 = h$ such that $h_t = \mathrm{id}$ on ∂_M for all t.

We say that M admits a ***compact exhaustion*** if there is a sequence of compact submanifolds with boundary $(M_i)_{i=1}^{\infty}$ with $\dim M_i = \dim M = n$ such that $M_1 \subset M_2^0 \subset M_2 \subset \dots$ and $M = \bigcup_{i=1}^{\infty} M_i$.

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Recall that a group G is ***perfect*** if $G = [G, G]$.

Some Basic Theorems

- **Theorem of Mather, 1971**
  The group $\mathcal{H}_c(\mathbb{R}^n)$ of all compactly supported homeomorphisms of $\mathbb{R}^n$ is perfect.
- **Theorem of McDuff, 1977**
  If M^0 is the interior of a compact manifold with boundary then the group $\mathcal{H}(M^0)$ is perfect.
- **Theorem of Fukui and Imanishi, 1996**
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Theorem

If M is compact or M admits a compact exhaustion, then the group $\mathcal{H}_c(M)$ is perfect. Moreover if $\partial_M \neq \emptyset$ and ∂_M is compact, then the group $\mathcal{H}_c^\partial(M)$ is perfect.

In contrast, for diffeomorphism groups we have the following

Remark

Let M be a smooth manifold with boundary and let $\mathcal{D}(M)$ be the identity component of the group of all compactly supported C^∞ diffeomorphisms of M. It was proved by Fukui that if $n = 1$ then the group $\mathcal{D}(M)$ is not perfect.

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PERFECTNESS

For an open $U \subset M$ let
$\mathcal{H}_U(M) = \{h \in \mathcal{H}_c(M) : \exists h_t : h_0 = \mathrm{id}, h_1 = h \text{ and } \forall t \operatorname{supp}(h_t) \subset U\}$.

Basic Lemma

Let $B \subset M$ be a ball (resp. a half-ball in the case that $\partial_M \neq \emptyset$ and $\dim M \geq 2$) and $U \subset M$ be an open subset such that $\overline{B} \subset U$. Then there are $\phi \in \mathcal{H}_U(M)$ and a homomorphism $S : \mathcal{H}_B(M) \to \mathcal{H}_U(M)$ such that $h = [S(h), \phi]$ for all $h \in \mathcal{H}_B(M)$. Moreover in the case that $\partial_M \neq \emptyset$ if $h \in \mathcal{H}_B(M)$ satisfies $h = \mathrm{id}$ on ∂_M then $S(h) = \mathrm{id}$ on ∂_M.

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FACTORIZATION

Let $G \leq \mathcal{H}(M)$ and let $\mathcal{B} = \{B \subset M : B \quad \text{is a ball or half-ball}\}$.

Definition

- G is called ***factorizable*** (resp. with respect to a covering $\mathcal{U} \subset \mathcal{B}$) if for every $g \in G$ there are $d \in \mathbb{N}$, $B_1 \dots B_d \in \mathcal{B}$ (resp. $B_1 \dots B_d \in \mathcal{U}$) and $g_1 \dots g_d \in G$ such that $g = g_1 \dots g_d \quad$ with $g_i \in G_{B_i}$ for all i.
- G is called ***locally continuously factorizable*** if for any finite subcovering $\mathcal{U} = (U_i)_{i=1}^d$ of $\mathcal{B}$ there exist a neighborhood $\mathcal{P}$ of $\mathrm{id} \in G$ and continuous mappings $\sigma_i : \mathcal{P} \to G$, $i = 1, \dots, d$, such that for all $f \in \mathcal{P}$ one has

$$f = \sigma_1(f) \dots \sigma_d(f), \quad \mathrm{supp}(\sigma_i(f)) \subset U_i, \forall i.$$

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If M is compact or it admits a compact exhaustion then the groups $\mathcal{H}_c(M)$ and $\mathcal{H}_c^\partial(M)$ are factorizable with respect to any covering $\mathcal{U} \subset \mathcal{B}$

Using results contained in the paper of Edwards and Kirby we can prove the following

Proposition

If M is compact then the groups $\mathcal{H}_c(M)$ and $\mathcal{H}_c^\partial(M)$ are locally continuously factorizable.

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Lemma

Suppose that B is a ball (or a half-ball and $\dim M \geq 2$), U is open in M and $\overline{B} \subset U$. Then $\mathcal{H}_B(M)$ is continuously perfect in $\mathcal{H}_U(M)$ with $r_{\mathcal{H}_B(M),\mathcal{H}_U(M)} = 1$.

Remark. The above lemma is no longer true in $\mathcal{C}^1$ cathegory.

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SIMPLICITY

Recall that a group G is ***simple*** if there are no nontrivial normal subgroups of G. Let $G \leq \mathcal{H}(M)$. We say that G is ***fixed point free*** (or *non-fixing*) if for every $x \in M$ there is $g \in G$ such that $g(x) \neq x$.

Proposition

If M is compact or it admits a compact exhaustion then there does not exist any fixed point free normal subgroup of $\mathcal{H}_c(M)$.

Corollary

Let M be connected. Assume that M is compact or it admits a compact exhaustion. Then $\partial_M = \emptyset$ iff $\mathcal{H}_c(M)$ is simple.

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If $\partial_M \neq \emptyset$ then $\mathcal{H}_c^{\partial}(M)$ is not simple.

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Recall that a group is ***bounded*** if it is bounded with respect to any bi-invariant metric.
Let G be a group. A ***conjugation-invariant norm*** on G (or *norm* for short) is a function $\nu : G \to [0, \infty)$ which satisfies following conditions. For any $g, h \in G$

- $\nu(g) > 0$ if and only if $g \neq e$;
- $\nu(g^{-1}) = \nu(g)$;
- $\nu(gh) \leq \nu(g) + \nu(h)$;
- $\nu(hgh^{-1}) = \nu(g)$.

It is easily seen that G is bounded if and only if any norm on G is bounded.

Recall that if M is compact or it admits a compact exhaustion then for every covering $\mathcal{U} \subset \mathcal{B}$ and for any $g \in \mathcal{H}_c(M)$ there are $g_1, \ldots, g_d$ and $U_1, \ldots U_d \in \mathcal{U}$ such that $g = g_1 \ldots g_d$ and for each i $g_i \in H_{U_i}(M)$.
For any $g \in \mathcal{H}_c(M), g \neq \mathrm{id}$ one can define the ***fragmentation norm*** denote by $\mathrm{frag}_M(g)$, which is the smallest d such that $g = g_1 \ldots g_d$ as above. By definition $\mathrm{frag}_M(\mathrm{id}) = 0$. Clearly frag_M is a norm on $\mathcal{H}_c(M)$.

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Theorem of Burago, Ivanov and Polterovich, 2008

Let $\mathrm{Diff}^\infty_c(M)$ denote the identity component of the group of all smooth compactly supported diffeomorphisms of M.

- The group $\mathrm{Diff}^\infty_c(S^n)$ is bounded, where S^n is a sphere.
- The group $\mathrm{Diff}^\infty_c(M)$ is bounded for any closed connected manifold M with $\dim M = 3$.
- The group $\mathrm{Diff}^\infty_c(M)$ is bounded iff frag_M is bounded.

Theorem

Let M be a compact manifold or M admits a compact exhaustion. Then $\mathcal{H}_c(M)$ is bounded if and only if frag_M is bounded.

Theorem

Let ∂_M be compact. If the group $\mathcal{H}_c(M)$ is bounded then $\mathcal{H}(\partial_M)$ is bounded as well. Moreover, if the group $\mathcal{H}_c(M^0)$ is bounded then so is the group $\mathcal{H}^\partial_c(M)$.

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Let ∂_M be compact. If the group $\mathcal{H}_c(M)$ is bounded then $\mathcal{H}(\partial_M)$ is bounded as well. Moreover, if the group $\mathcal{H}_c(M^0)$ is bounded then so is the group $\mathcal{H}_c^\partial(M)$.

Theorem of Burago, Ivanov and Polterovich, 2008
Let $\mathrm{Diff}_c^\infty(M)$ denote the identity component of the group of all smooth compactly supported diffeomorphisms of M.

- The group $\mathrm{Diff}_c^\infty(S^n)$ is bounded, where S^n is a sphere.
- The group $\mathrm{Diff}_c^\infty(M)$ is bounded for any closed connected manifold M with $\dim M = 3$.
- The group $\mathrm{Diff}_c^\infty(M)$ is bounded iff frag_M is bounded.

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Definition

A connected open manifold M is called ***portable (in the wider sense)*** *if there are disjoint open subsets U, V of M such that there is $f \in \mathcal{H}_c(M)$ with $\overline{f(U \cup V)} \subset V$. Furthermore, for every compact subset $K \subset M$ there is $h \in \mathcal{H}_c(M)$ satisfying $h(K) \subset U$.*

Some Basic Theorems

- **Theorem of Burago, Ivanov and Polterovich, 2008**
  If M is portable then the group $\mathrm{Diff}^{\infty}_c(M)$ is bounded.
- **Theorem of Rybicki 2011**
  Assume that M^0 is an interior of a compact manifold M and M^0 is portable. Let $\mathcal{D}^r(M^0)$ for $r = 0, \ldots, \infty$ denote the identity component of the group of all $\mathcal{C}^r$ diffeomorphisms of M^0 which can be joined with the identity by compactly supported isotopies. Then the group $\mathcal{D}^r(M^0)$ is bounded.

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BOUNDEDNESS

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If M is portable then $\mathcal{H}_c(M)$ is bounded. If M^0 is portable then $\mathcal{H}_c^\partial(M)$ is bounded. In particular, $\mathcal{H}_c(\mathbb{R}^n)$ is bounded.

In contrast, for diffeomorphism groups we have the following

Proposition

Let M be a smooth manifold with boundary and let $\mathcal{D}(M)$ be the identity component of the group of all compactly supported C^∞ diffeomorphisms of M. Let $\mathcal{D}^\partial(M)$ denote the subgroup of $\mathcal{D}(M)$ consists of all $f \in \mathcal{D}(M)$ such that there is a compactly supported isotopy f_t with $f_0 = \mathrm{id}$ and $f_1 = f$ satisfying $f_t|_{\partial(M)} = \mathrm{id}$ for all t. Then $\mathcal{D}^\partial(M)$ is an unbounded group.

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BIBLIOGRAPHY

- Burago D., Ivanov S., Polterovich S., Conjugation invariant norms on groups of geometric origin, Adv. Stud. Pure Math., 2008, 52, Groups of Diffeomorphisms, 221-250.
- Edwards R.D., Kirby R.C., Deformations of spaces of imbeddings, Ann. Math., 1971, 93, 63-88.
- Fukui K., Imanishi H., On commutators of foliation preserving homeomorphisms, J. Math. Soc. Japan, 1999, 51, No. 1, 227-236.
- Mather J.N., The vanishing of the homology of certain groups of homeomorphisms, Topology, 1971, 10, 297-298.
- McDuff D., The lattice of normal subgroups of the group of diffeomorphisms or homeomorphisms of an open manifold, J. Lond. Math. Soc. (2), 1978, 18, 353-364.
- Rybicki T., Boundedness of certain automorphism groups of an open manifold, Geom. Dedicata, 2011, 151,1, 175-186.